

On dS_4 extremal surfaces and entanglement entropy in some ghost CFTs

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Abstract

In arXiv:1501.03019 [hep-th], the areas of certain complex extremal surfaces in de Sitter space were found to have resemblance with entanglement entropy in appropriate dual Euclidean non-unitary CFTs, with the area being real and negative in dS_4 . In this paper, we study some toy models of 2-dim ghost conformal field theories with negative central charge with a view to exploring this further from the CFT point of view. In particular we consider bc -ghost systems with central charge $c = -2$ and study the replica formulation for entanglement entropy for a single interval, and associated issues arising in this case, notably pertaining to (i) the $SL(2)$ vacuum coinciding with the ghost ground state, and (ii) the background charge inherent in these systems which leads to particular forms for the norms of states (involving zero modes). This eventually gives rise to negative entanglement entropy. We also discuss a (logarithmic) CFT of anti-commuting scalars, with similarities in some features. Finally we discuss a simple toy model of two “ghost-spins” which mimics some of these features.

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1 Introduction

dS/CFT duality [1, 2, 3] involves certain generalizations of gauge/gravity duality [4, 5, 6, 7] conjecturing that de Sitter space is dual to a hypothetical Euclidean non-unitary CFT that lives on the future boundary \mathcal{I}^+ . More precisely the late-time wavefunction of the universe Ψ_{dS} with appropriate boundary conditions is equated with the dual CFT partition function Z_{CFT} [3]. The dual CFT_d energy-momentum tensor correlator $\langle TT \rangle$ obtained in a semiclassical approximation $\Psi \sim e^{iS}$ exhibits central charge coefficients $\mathcal{C}_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$ in dS_{d+1} . In dS_4 , the central charge is real and negative: the dual CFT is thus reminiscent of ghost-like non-unitary theories. See *e.g.* [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] for further work on dS/CFT : in particular in [12], a higher spin dS_4 duality was conjectured involving a 3-dim CFT of anti-commuting $Sp(N)$ (ghost) scalars.

Regardless of various details, it is perhaps of interest to better organize our understanding of de Sitter space using this dual conformal invariance, as well as constrain the properties such a hypothetical CFT might have. In this context, it is perhaps of interest to study entanglement entropy as a probe of dS/CFT . Some attempts in this regard were begun in [19, 20] by studying certain generalizations of the Ryu-Takayanagi prescription [21, 22] in AdS/CFT . We recall that in the gravity approximation, the areas in Planck units of bulk minimal surfaces (more generally extremal surfaces [23]) anchored at the subsystem interface and dipping into the bulk capture entanglement entropy of the field theory subsystem (see [24, 25] for reviews). A different perspective on this appears more recently in [26] (see also [27, 28]).

In the de Sitter context, for strip-shaped subregions on a constant Euclidean time slice of the future boundary, it was found in [19] that while real surfaces that might be of relevance have vanishing area and are thus uninteresting, the areas of certain codim-2 complex extremal surfaces have structural resemblance with entanglement entropy of dual Euclidean CFTs (reviewed in sec. 2). The coefficients of the leading divergent “area law” terms in dS_{d+1}

resemble the central charges $\mathcal{C}_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$ of the CFT_{dS} appearing in the $\langle TT \rangle$ correlators in [3]. In general the areas thus obtained are negative or pure imaginary. In particular, in dS_4 , where the dual CFT central charge is $-\frac{R_{dS}^2}{G_4}$, the area is negative. (This leading divergence has also been studied in [29].) In [20], this was generalized to spherical subregions on a constant Euclidean time slice of the future boundary. In this case, in even boundary dimensions, the area exhibits a subleading logarithmic divergence: the coefficient of this piece matches precisely including numerical factors with the coefficient of the subleading logarithmic divergence in the free energy of the CFT on a sphere, using the dS/CFT dictionary $Z_{\text{CFT}} \equiv e^{-F} = \Psi_{dS} \sim e^{iS}$ with Ψ_{dS} the wavefunction of the universe in a classical approximation. In [18], certain asymptotically de Sitter spaces were argued to be gravity duals of the CFT at uniform energy density, and thus are analogs of black branes: similar complex extremal surfaces in these backgrounds exhibit a finite cutoff-independent extensive part in the areas, analogous to thermal entropy, but again negative for the dS_4 black brane [19]. The resulting analysis in all these cases ends up being equivalent to analytic continuation from the Ryu-Takayanagi expressions in AdS/CFT .

Towards regarding the areas of these complex extremal surfaces as entanglement entropy, it is of interest to focus on the dS_4 case where the central charge and areas are negative, and look for toy models where negative entanglement entropy arises, if at all. It is then natural to look to non-unitary Euclidean CFTs with negative central charge, in particular in the context of 2-dim CFTs where conformal symmetry is well-known to be especially powerful [30]. In this work, we revisit the replica formulation [31, 32, 33] of entanglement entropy for a single interval, in certain 2-dim ghost conformal field theories which have negative central charge as is well-known. Focussing first on the bc -ghost system, sec. 3 (see *e.g.* [34, 35], as well as [36], and more recently [37, 38, 39, 40, 41]), we note that the $SL(2)$ vacuum is in general not equivalent to the ground state: for the $c = -2$ bc -ghost system, the two coincide. In this case we find that the replica formulation is formally applicable in the $SL(2)$ invariant vacuum, with the twist field operators exhibiting negative conformal dimensions leading to negative entanglement entropy (a study of a \mathbb{Z}_N orbifold of the bc -ghost CFT corroborates the negative dimension of the twist field operators). A crucial ingredient is that the “inner product” required for nonvanishing observables reflects the charge asymmetry due to the background charge inherent in these ghost systems: this is equivalent to the presence of ghost zero mode insertions which soak up the background charge. It is under these conditions that correlation functions such as $\langle bc \rangle$ are nonvanishing. The replica formulation then applies and leads to nonvanishing negative entanglement entropy.

Then (in sec. 4) we discuss a Euclidean 2-dim conformal field theory of complex anti-commuting ghost scalars $\chi, \bar{\chi}$. This is a logarithmic CFT, discussed in *e.g.* [42, 43, 44, 38, 39,

40, 41]. This is in part motivated by the $Sp(N)$ higher spin 3-dim theory of anti-commuting scalars conjectured to be dual to dS_4 in [12] (studied previously in [45, 46]). Although one might expect logarithms in correlation functions, the object of interest for a single interval is a 2-point function of twist operators which exhibits power law behaviour with no logarithms. This then gives the same form as above for entanglement entropy. In sec. 5, we discuss a toy model of two “ghost spins” with a non-positive inner product which mimics some of the features of the ghost systems above: the reduced density matrix obtained thus has some negative eigenvalues and formally gives entanglement entropy for some states with negative real part as well as an imaginary part. Sec. 6 has a discussion in part on the negative areas of the complex dS_4 extremal surfaces and negative EE, while Appendix A reviews some details on the replica calculation.

It is important to note that the analysis here is in toy 2-dim ghost CFTs with negative central charge and simply serves to illustrate that negative EE can formally be obtained from appropriate generalizations of the standard CFT replica technique. The resulting object, being negative, does not satisfy properties of ordinary EE such as strong subadditivity (Sec. 6). These 2-dim toy ghost CFTs here are motivated by the 3-dim $Sp(N)$ higher spin theory in [12] (and are not to be considered as dual to dS_3 which has imaginary central charge): it would be interesting to see if the present work can be used to gain insights into the duals to dS_4 and dS_3 .

2 Reviewing de Sitter extremal surfaces

Here we briefly review the study [19, 20] of bulk extremal surfaces anchored over strip- or sphere-shaped subregions on the future boundary \mathcal{I}^+ of de Sitter space dS_{d+1} in the Poincare slicing or planar coordinate foliation. The metric is

$$ds^2 = \frac{R_{dS}^2}{\tau^2}(-d\tau^2 + dw^2 + d\sigma_{d-1}^2) , \quad (1)$$

covering half of the spacetime, *e.g.* the upper patch, with \mathcal{I}^+ at $\tau = 0$ and a coordinate horizon at $\tau = -\infty$. This may be obtained by analytic continuation of a Poincare slicing of AdS ,

$$z \rightarrow -i\tau , \quad R_{AdS} \rightarrow -iR_{dS} , \quad t \rightarrow -iw , \quad (2)$$

where w is akin to boundary Euclidean time, continued from time in AdS (with z the bulk coordinate). The dual Euclidean CFT is taken as living on the future $\tau = 0$ boundary \mathcal{I}^+ . We assume translation invariance with respect to the boundary Euclidean time direction w , and consider a subregion on a $w = \text{const}$ slice of \mathcal{I}^+ . One might imagine that tracing out the complement of this subregion then gives entropy in some sense stemming from the

information lost. In the bulk, we mimic this by appropriate de Sitter extremal surfaces on the $w = \text{const}$ slice, analogous to the Ryu-Takayanagi prescription in AdS/CFT . Operationally these extremal surfaces begin at the interface of the subsystem (or subregion) and dip into the bulk time direction.

For a strip-shaped subregion on \mathcal{I}^+ (with width say along x), parametrizing the spatial part in (1) as $d\sigma_{d-1}^2 = \sum_{i=1}^{d-1} dx_i^2$ with $x \in \{x_i\}$, the dS_{d+1} area functional on a $w = \text{const}$ slice is $S_{dS} = \frac{R_{dS}^{d-1} V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\left(\frac{dx}{d\tau}\right)^2 - 1}$. After extremization this gives $\dot{x}^2 = \frac{-A^2 \tau^{2d-2}}{1 - A^2 \tau^{2d-2}}$, with $\frac{dx}{d\tau} \equiv \dot{x}$, and A^2 is a conserved constant. Due to a crucial minus sign relative to the AdS case, real surfaces (obtained with $A^2 < 0$) do not exhibit any turning point (where $|\dot{x}| \rightarrow \infty$). Developing this further, it turns out that these give null surfaces with vanishing area, uninteresting from the point of view of entanglement entropy. On the other hand, it can be shown that certain complex extremal surfaces can be identified if the bulk time parameter takes an imaginary path $\tau = iT$. These are parametrized as $x(\tau)$, with $-\dot{x}^2 = \left(\frac{dx}{dT}\right)^2 = \frac{A^2 (-1)^{d-1} T^{2d-2}}{1 - (-1)^{d-1} A^2 T^{2d-2}}$. These complex solutions to the extremization problem exist if the constant parameter A^2 satisfies $A^2 > 0$ for dS_4 (and odd d more generally), and $A^2 < 0$ for dS_3, dS_5 (and even d). These are smooth surfaces exhibiting a turning point $T_*^{d-1} = \frac{1}{A}$ where $|\frac{dx}{dT}| \rightarrow \infty$. Thus for a strip subregion, the area has the form

$$S_{dS} = -i \frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1 - (-1)^{d-1} A^2 \tau^{2d-2}}} = i^{1-d} \frac{R_{dS}^{d-1}}{2G_{d+1}} V_{d-2} \left(\frac{1}{\epsilon^{d-2}} - c_d \frac{1}{l^{d-2}} \right), \quad (3)$$

where $\tau_{UV} = i\epsilon$ and $\tau_* = il$, and the integral ends up being similar to that in AdS (with corresponding constant c_d). The area of these surfaces passes several checks from the point of view of regarding these as entanglement entropy in the dual Euclidean CFT in a dS/CFT perspective. S_{dS} in (3) bears structural resemblance to entanglement entropy in a dual CFT with central charge $\mathcal{C}_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$. The first term $S_{dS}^{div} \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}} \frac{V_{d-2}}{\epsilon^{d-2}}$ resembles an area law divergence [47, 48], proportional to the area of the interface between the subregion and the environment, in units of the ultraviolet cutoff. It appears independent of the shape of the subregion, expanding the area functional and assuming that \dot{x} is small near the boundary τ_{UV} . Rewriting this as $\mathcal{C}_d \frac{V_{d-2}}{\epsilon^{d-2}}$, we see that it is also proportional to the central charge $\mathcal{C}_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$ representing in some sense the number of degrees of freedom in the dual (non-unitary) CFT: these arose in the $\langle TT \rangle$ correlators obtained in [3]. In dS_4 , the central charge $\mathcal{C} \sim -\frac{R_{dS}^2}{G_4}$ is real and negative, while in dS_3, dS_5 , it is imaginary. The second term is a finite cutoff-independent piece.

Likewise for a spherical subregion on a constant boundary Euclidean time slice $w = \text{const}$, there are complex extremal surfaces of the form $r(\tau) = \sqrt{l^2 + \tau^2}$ with $\tau = iT$. (There are also real null surfaces with vanishing area, as well as those of the form above, for τ real, with

no finite cutoff-independent pieces.) Then for even boundary dimensions d , the area of these complex extremal surfaces exhibits a subleading logarithmic divergence whose coefficient is related to the trace anomaly of the dual Euclidean CFT_d . This can be seen to match precisely with the coefficient of a corresponding logarithmic divergence in the free energy of the CFT on a sphere, obtained via the dS/CFT dictionary with the wavefunction of the universe in a classical approximation $Z_{\text{CFT}} = e^{-F} = \Psi_{dS} \sim e^{iS_{cl}}$.

The resulting expressions obtained in the end amount to analytic continuation from the Ryu-Takayanagi expressions [21, 22] for holographic entanglement entropy in AdS/CFT , although this was not obvious to begin with starting directly in de Sitter space.

4-dim de Sitter space, dS_4 : For a strip subregion of width l on the future boundary \mathcal{I}^+ of de Sitter space dS_4 restricting to a constant boundary Euclidean time slice, the complex extremal surfaces have area

$$S_{dS} \sim -\frac{R_{dS}^2}{G_4} \left(\frac{V_1}{\epsilon} - \frac{V_1}{l} \right) \quad (4)$$

following from (3), while for a spherical subregion of radius l , we have $S_{dS} = -\frac{\pi R_{dS}^2}{2G_4} (\frac{l}{\epsilon} - 1)$. The finite constant cutoff-independent piece $\frac{\pi R_{dS}^2}{2G_4}$ is a universal term, positive definite (note that it resembles de Sitter entropy). Thus for dS_4 , we see that the areas of these complex extremal surfaces are real and negative. Structurally they resemble entanglement entropy in a 3d CFT with negative central charge.

Although we have obtained complex surfaces as solutions to this extremization question, the resulting expressions are all real: specifically with $\tau = iT$, we rewrite the expressions in terms of $T = |\tau|$, obtaining

$$\begin{aligned} \frac{\Delta x}{2} = \frac{l}{2} &= \int_0^{T_*} \frac{(T^2/T_*^2) dT}{\sqrt{1 - (T^4/T_*^4)}} \equiv \int_0^{|\tau_*|} \frac{(\frac{|\tau|^2}{|\tau_*|^2}) d|\tau|}{\sqrt{1 - \frac{|\tau|^4}{|\tau_*|^4}}}, \\ S_{dS_4} &= -i \frac{R_{dS}^2}{4G_4} V_1 \int_{\tau_{UV}}^{\tau_*} \frac{d\tau}{\tau^2} \frac{1}{\sqrt{1 - \tau^4/\tau_*^4}} = -\frac{R_{dS}^2}{4G_4} V_1 \int_{|\tau_{UV}|}^{|\tau_*|} \frac{d|\tau|}{|\tau|^2} \frac{1}{\sqrt{1 - \frac{|\tau|^4}{|\tau_*|^4}}}. \end{aligned} \quad (5)$$

These resulting expressions are of course related in a very simple way to the Ryu-Takayanagi AdS expressions. The point of this rewriting is to make manifest the mapping from the complex τ -path to corresponding real τ -values in dS_4 . For each $\tau = iT$ ranging from τ_{UV} to τ_* , we have a corresponding real τ given simply by $\tau_R \equiv -T = -|\tau|$, ranging over $(-\epsilon, -T_*)$ in the bulk de Sitter space. The strip width l is then related to this bulk dS_4 time τ_R as $l \sim T_*$: thus increasing l means larger $|\tau_R|$, *i.e.* further back in the past. These expressions while real-valued are of course completely different from the extremization process restricting to real τ -values in dS_4 [19]: the latter lead to null surfaces with vanishing area and thus no bearing on entanglement entropy.

The nonunitary dual CFTs in the de Sitter context, with negative or imaginary central charge, contain operators with complex conformal dimensions in general. For instance, bulk scalar modes of mass m in dS_{d+1} obey the wave equation $\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi) - m^2\phi = 0$. This becomes $\tau^2\ddot{\varphi} - (d-1)\tau\dot{\varphi} + (k^2\tau^2 + m^2R_{dS}^2)\varphi = 0$ for modes of the form $\varphi(\tau)e^{ik_ix^i}$. Near the future boundary $\tau \rightarrow 0$, these are approximated as $\varphi \sim \tau^\Delta$, giving the dual conformal dimensions $\Delta(\Delta - d) = -m^2R_{dS}^2$ *i.e.* $\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - m^2R_{dS}^2}$ (analogous to AdS/CFT). So for bulk de Sitter modes with sufficiently large mass, the dual operator conformal dimensions Δ are complex: $\Delta = \frac{d}{2} \pm i\sqrt{m^2R_{dS}^2 - \frac{d^2}{4}} \sim \frac{d}{2} \pm imR_{dS}$ for $mR_{dS} \gg 1$. This is a feature of dS/CFT and more generally of field asymptotics in de Sitter space (in the higher spin dS_4/CFT_3 case [12], the conformal dimensions are real-valued). However we note that for all $mR_{dS} \gtrsim \frac{d}{2}$, the real part of the conformal dimensions is fixed, and positive with $\text{Re}\Delta = \frac{d}{2}$. For bulk modes with $mR_{dS} \leq \frac{d}{2}$, the conformal dimensions are real and positive. Thus in the approximation of bulk de Sitter theories consistently truncated to only modes with masses satisfying $mR_{dS} \leq \frac{d}{2}$, all dual conformal dimensions are real and with $O(1)$ values.

It is worth noting that these complex solutions lie outside the original de Sitter coordinate range (where τ is real) so one may conclude that, strictly speaking, there are no extremal surfaces with nonvanishing area, and correspondingly no notion of dual entanglement entropy in de Sitter space in the sense here, based on the Ryu-Takayanagi formulation (also the dual theory is Euclidean so the boundary Euclidean time slice used here may be regarded as ad hoc). If instead we consider the complex extremal surfaces, focussing on dS_4 , we have seen negative areas (and thereby negative EE). Towards understanding this from a CFT point of view, we will study some toy models of 2-dim CFTs with negative central charge and entanglement entropy from a replica formulation. Specifically we focus on some 2-dim ghost CFTs, in particular the bc -ghost system and the CFT of anti-commuting ghost scalars.

3 bc -ghost CFTs

The bc -ghost CFT is familiar from worldsheet string theory: for our present purposes, see *e.g.* [34, 35], as well as [36], and more recently [37, 38, 39, 40, 41]. Our discussion here is to some extent simply a review of some relevant aspects, but adapted to the present context. The action and conformal weights

$$S = \frac{1}{2\pi} \int d^2z \, b\bar{\partial}c, \quad h_b = \lambda, \quad h_c = 1 - \lambda, \quad (6)$$

for any λ ensure conformal invariance in this holomorphic part of the CFT. The conformal fields b, c , are anticommuting variables. (We will suppress writing the anti-holomorphic parts in most of our discussion.) We have the equations of motion $\bar{\partial}c(z) = 0 = \bar{\partial}b(z)$, and

$\bar{\partial}b(z)c(w) = 2\pi\delta^2(z-w, \bar{z}-\bar{w})$. This gives the singular parts of the OPEs

$$b(z)c(w) \sim \frac{1}{z-w}, \quad c(z)b(w) \sim \frac{1}{z-w}, \quad (7)$$

and the energy-momentum tensor and central charge

$$T(z) = :(\partial b)c: - \lambda \partial(:bc:) = \frac{1}{2}(:(\partial b)c: - :b\partial c:) + \frac{1}{2}Q\partial(:bc:), \quad (8)$$

$$c = 1 - 3(2\lambda - 1)^2 = 1 - 3Q^2, \quad Q = 1 - 2\lambda. \quad (9)$$

The central charge can be seen from the TT OPE which is of standard form. Q is the background charge for the ghost system and is nonvanishing except when $\lambda = \frac{1}{2}$ (which corresponds to two $c = \frac{1}{2}$ free fermions). The mode expansions can be written as

$$b(z) = \sum_{m \in \mathbb{Z}} \frac{b_m}{z^{m+\lambda}}, \quad c(z) = \sum_{m \in \mathbb{Z}} \frac{c_m}{z^{m+1-\lambda}}. \quad (10)$$

Inverting we have $b_m = \oint \frac{dz}{2\pi i} z^{m+\lambda-1} b(z)$, $c_m = \oint \frac{dz}{2\pi i} z^{m-\lambda} c(z)$. These and the OPEs lead via the contour arguments to the oscillator algebra and Virasoro generators

$$\begin{aligned} \{b_m, c_n\} &= \delta_{m+n,0}, \quad \{b_m, b_n\} = 0, \quad \{c_m, c_n\} = 0, \\ L_m &= \sum_{n=-\infty}^{\infty} (m\lambda - n)b_n c_{m-n} \quad [m \neq 0]; \quad L_0 = \sum_{n>0} n(b_{-n}c_n + c_{-n}b_n) + \frac{\lambda(1-\lambda)}{2}, \end{aligned} \quad (11)$$

which satisfy the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}$, where $A(m)$ is a central term. Since b, c are real fields, the hermiticity relations are $b_m^\dagger = b_{-m}$, $c_m^\dagger = c_{-m}$, and $L_m^\dagger = L_{-m}$, $L_0^\dagger = L_0$. The zero mode sector $\{b_0, c_0\} = 1$ gives two states $|\downarrow\rangle, |\uparrow\rangle$, with

$$b_0|\downarrow\rangle = 0, \quad c_0|\downarrow\rangle = |\uparrow\rangle, \quad b_0|\uparrow\rangle = |\downarrow\rangle, \quad c_0|\uparrow\rangle = 0, \quad (12)$$

and all b_n, c_n , $n > 0$ annihilating both $|\downarrow\rangle, |\uparrow\rangle$. It is conventional to group b_0 with the annihilation operators and take $|\downarrow\rangle$ as the ghost ground state.

We define the $SL(2, \mathbb{Z})$ invariant vacuum $|0\rangle$ as the vacuum where the energy-momentum tensor and the conformal fields are regular at the origin $z = 0$ of the z -plane:

$$|0\rangle : \quad T(z)|0\rangle = \sum_m \frac{L_m}{z^{m+2}}|0\rangle = \text{regular} \quad \Rightarrow \quad L_m|0\rangle = 0, \quad m \geq -1. \quad (13)$$

Thus $L_0, L_{\pm 1}$ annihilate $|0\rangle$. In particular $L_0|0\rangle = 0$ implies that the $SL(2)$ vacuum has zero L_0 eigenvalue. Similarly, using the mode expansions (10) above, we see that regularity of $b(z)|0\rangle$ and $c(z)|0\rangle$ at $z = 0$ gives

$$|0\rangle : \quad b_{m \geq 1-\lambda}|0\rangle = 0, \quad c_{m \geq \lambda}|0\rangle = 0. \quad (14)$$

The ghost ground state, of lowest L_0 eigenvalue, does not necessarily coincide with the $SL(2)$ vacuum $|0\rangle$ however, due to the shifts in the mode expansions (10). In particular, $L_0|\downarrow\rangle = \frac{\lambda(1-\lambda)}{2}|\downarrow\rangle$ so that for $\lambda > 1$, the L_0 eigenvalue of $|\downarrow\rangle$ is negative.

The $U(1)$ charge symmetry $\delta b = -i\epsilon b$, $\delta c = i\epsilon c$ gives the ghost current $j(z) = -:bc:$, with the OPEs $j(z)b(w) \sim -\frac{1}{z-w}b(w)$, $j(z)c(w) \sim \frac{1}{z-w}c(w)$, and $j(z)j(w) \sim \frac{1}{(z-w)^2}$. The Tj OPE exhibits an anomalous transformation

$$T(z)j(w) \sim \frac{Q}{(z-w)^3} + \frac{1}{(z-w)^2}j(w) + \frac{1}{z-w}\partial j(w), \quad (15)$$

the non-tensor term in the OPE arising from the background charge $Q = 1 - 2\lambda$. This leads to a finite transformation $(\partial_z w)j'(w) = j(z) - \frac{Q}{2}\frac{\partial_z^2 w}{\partial_z w}$. The ghost number N_g on the cylinder and that on the plane N_g^z are

$$N_g = \int_0^{2\pi} \frac{dw}{2\pi i} j^{cyl}(w) = \sum_{n=1}^{\infty} (c_{-n}b_n - b_{-n}c_n) + c_0b_0 - \frac{1}{2}, \quad N_g^z = \oint \frac{dz}{2\pi i} j(z) = N_g - \frac{Q}{2}. \quad (16)$$

The ghost number on the plane N_g^z , counting j_0 charge, is obtained from the finite transformation above for the cylinder coordinate $w = \log z$, the anomalous term leading to the shift. We see that $[N_g, b_m] = -b_m$, $[N_g, c_m] = c_m$. Conventionally, the ghost number of states is taken as $N_g|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$, $N_g|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$. This gives $N_g^z|\downarrow\rangle = -\frac{Q+1}{2}|\downarrow\rangle = (\lambda-1)|\downarrow\rangle$, and implies that $N_g^z|0\rangle = 0$. Since the $SL(2)$ vacuum $|0\rangle$ satisfies (13), (14), the b_m operators for $1-\lambda \leq m < 0$ act as annihilation operators on $|0\rangle$ although they are creation operators for $|\downarrow\rangle$. This implies that the $SL(2)$ vacuum interpreted as a Fermi sea is filled for the $b_{1-\lambda \leq m < 0}$ oscillators, and we can identify this as

$$|0\rangle = b_{-1}b_{-2}\dots b_{1-\lambda}|\downarrow\rangle = \prod_{1-\lambda \leq m < 0} b_m|\downarrow\rangle \quad [\lambda \neq 1]. \quad (17)$$

This then implies that

$$N_g|0\rangle = \left(-\frac{1}{2} - (\lambda-1)\right)|0\rangle = \frac{Q}{2}|0\rangle, \quad i.e. \quad N_g^z|0\rangle = 0, \quad (18)$$

since the b -oscillators cancel the N_g^z ghost charge $\lambda-1$ of $|\downarrow\rangle$. In other words, the $SL(2)$ vacuum $|0\rangle$ has zero j_0 charge on the plane.

The presence of the background charge Q forces a particular form for the adjoints of states and correspondingly their norms. The OPE (15) and $j(z) = \sum_m \frac{j_m}{z^{m+1}}$ gives

$$[L_m, j_n] = -nj_{m+n} + \frac{1}{2}Qm(m+1)\delta_{m,-n}. \quad (19)$$

In particular, while $j_n^\dagger = -j_n$ for $n \neq 0$ (with $j_n = -\sum_m b_m c_{n-m}$), the asymmetry between $m = +1$ and $m = 0, -1$, above gives

$$[L_1, j_{-1}] = j_0 + Q, \quad [L_1, j_{-1}]^\dagger = [L_{-1}, j_1] = -j_0 \quad \Rightarrow \quad j_0^\dagger = -(j_0 + Q), \quad (20)$$

so that there is an asymmetry in the bc -system from the charge operator j_0 , indicating the presence of the background charge Q . In particular charge neutral operators have vanishing correlation functions: with $[j_0, O_p] = pO_p$, and $j_0|q\rangle = q|q\rangle$, we have

$$p\langle q'|O_p|q\rangle = \langle q'|[j_0, O_p]|q\rangle = (-q' - Q - q)\langle q'|O_p|q\rangle, \quad (21)$$

so that the correlation function is non-vanishing only if $p = -(q + q' + Q)$. Considering $p = 0$, we have $q' = -q - Q$ for a nonvanishing correlation function, so that the inner products of the states $|q\rangle$ can be normalized as

$$\langle -q - Q|q\rangle = 1. \quad (22)$$

Since nonzero correlation functions only arise after inserting operators that soak up the background charge Q , this normalization can be interpreted as the inner product of the $|q\rangle$ in-state with an out-state $\langle -q - Q|$ which is the adjoint of $|q\rangle$. In particular for the $SL(2)$ vacuum $|0\rangle$ taken as the in-state, we require the out-state $\langle -Q|$ for a nonvanishing correlator. Recalling that the difference in the number of zero modes is $\#_0^c - \#_0^b = -\frac{1}{2}Q\chi = -Q$ on the sphere (with Euler number $\chi = 2$), we see that the normalization (22) precisely corresponds to inserting the appropriate number of operators that soak up the zero modes of the bc -ghost system in order to obtain nonvanishing correlation functions. From a path integral point of view, we see that the vacuum partition function $\langle 0|0\rangle \equiv \int DbDc e^{-S}$ vanishes due to Grassmann integration over unbalanced zero modes of the b, c -fields (which do not appear in the action). The requisite number of zero mode insertions makes this nonvanishing and amounts to the normalization (22).

To illustrate this, consider the $\lambda = 2$ bc -system, with background charge $Q = -3$ and central charge $c = -26$, which arises in the reparametrization ghosts of the string worldsheet theory. The (b, c) -fields have conformal weights $(h_b, h_c) = (2, -1)$. The $SL(2)$ invariant vacuum $|0\rangle$ satisfies $b_{m \geq -1}|0\rangle = 0$, $c_{m \geq 2}|0\rangle = 0$, and we identify $|0\rangle \equiv b_{-1}|\downarrow\rangle$. In this case, we have $L_0|\downarrow\rangle = -|\downarrow\rangle$. We see that $\langle \downarrow | \downarrow \rangle = \langle \downarrow | \{b_0, c_0\} | \downarrow \rangle = 0$ since b_0 and $b_0^\dagger = b_0$ annihilate the states. Since $Q = -3$, the normalization (22) becomes $\langle +3|0\rangle = 1$ for the $SL(2)$ vacuum. The difference in zero modes is $\#_0^c - \#_0^b = -\frac{1}{2}Q\chi = 3$ on the sphere, and the smallest nonvanishing correlation function (which is not annihilated by inserting the oscillator relation) requires the insertion of three c -operators, *e.g.* $\langle 0|c_{-1}c_0c_1(\dots)|0\rangle$, as is well-known from worldsheet string amplitudes.

3.1 bc -ghosts with $\lambda = 1$

In this case, the conformal weights of the (b, c) -fields, the background charge Q and the central charge c are

$$\lambda = 1 : \quad (h_b, h_c) = (1, 0), \quad Q = -1, \quad c = -2. \quad (23)$$

The mode expansions (10) in this case are

$$b(z) = \sum_m \frac{b_m}{z^{m+1}} , \quad c(z) = \sum_m \frac{c_m}{z^m} . \quad (24)$$

From (14), the $SL(2)$ invariant vacuum satisfies

$$|0\rangle : \quad b_{m \geq 0}|0\rangle = 0 , \quad c_{m \geq 1}|0\rangle = 0 , \quad (25)$$

which coincide with the conditions (12) defining the ghost ground state $|\downarrow\rangle$. Thus instead of (17), we identify

$$\lambda = 1 : \quad |0\rangle = |\downarrow\rangle , \quad (26)$$

i.e. the $SL(2)$ invariant vacuum $|0\rangle$ coincides with the ghost ground state $|\downarrow\rangle$. To see consistency of this, we note from (18) that $N_g|0\rangle = -\frac{1}{2}|0\rangle$, which coincides with the ghost number of $|\downarrow\rangle$: also (16) gives $N_g^z|\downarrow\rangle = (\lambda - 1)|\downarrow\rangle = 0$. The Virasoro generators (11) become

$$L_m = \sum_{n=-\infty}^{\infty} (m-n)b_n c_{m-n} \quad [m \neq 0] ; \quad L_0 = \sum_{n>0} n(b_{-n}c_n + c_{-n}b_n) , \quad (27)$$

so that $L_0|\downarrow\rangle = L_0|0\rangle = 0$. Thus in this case, the $SL(2)$ vacuum is also the ground state of the system with $L_0 = 0$.

The field of lowest conformal dimension is $c(z)$ with $h_c = 0$. However since this is a first order system of anticommuting fields, the b, c , OPEs are not logarithmic, with the cc OPE vanishing. c_0 does not annihilate $|0\rangle$: instead $c_0|0\rangle = |\uparrow\rangle$. The state $c_0|0\rangle$ with $L_0 = 0$ is degenerate with $|0\rangle$. The ghost numbers are $N_g c_0|0\rangle = \frac{1}{2}c_0|0\rangle$ and $N_g^z c_0|0\rangle = c_0|0\rangle$.

The background charge here is $Q = -1$ and the normalization (22) in this case becomes $\langle -q+1|q\rangle = 1$: for the $SL(2)$ vacuum, this is $\langle +1|0\rangle = 1$. Using (24) and (25), the smallest nonvanishing correlation function is

$$\langle 0|c(z)|0\rangle = \langle 0|\sum_m \frac{c_m}{z^m}|0\rangle = \langle 0|c_0|0\rangle \equiv \langle +1|0\rangle ; \quad \langle +1| = \langle 0|c_0 , \quad (28)$$

so that the adjoint of the $SL(2)$ vacuum is the state $\langle 0|^\dagger = \langle 0|c_0$, with $c_0^\dagger = c_0$. More general nonvanishing correlation functions in the $|0\rangle$ in-state are obtained as above, with the out-state $\langle 0|c_0$. For instance

$$\langle b(z)c(w)\rangle_0 \equiv \langle 0|c_0 \sum_{m,n} \frac{b_m}{z^{m+1}} \frac{c_n}{w^n} |0\rangle = \langle 0|c_0 \sum_{m=0}^{\infty} \frac{w^m}{z^{m+1}} b_m c_{-m} |0\rangle = \frac{1}{z} \frac{1}{1 - \frac{w}{z}} \langle 0|c_0|0\rangle = \frac{1}{z-w} , \quad (29)$$

which is the expected form of the 2-point function. Note that without the c_0 -insertion, we have $\langle 0|b(z)c(w)|0\rangle = \frac{1}{z-w}\langle 0|0\rangle = 0$, using *e.g.* arguments such as (21). Thus the short distance structure of the bc -ghost system is only visible in correlation functions with insertions that appropriately cancel the background charge.

With the norms of states defined in this manner, we see that there are various negative norm states in this system, as expected from the negative central charge. Among the simplest is *e.g.* $(b_{-1} - c_{-1})|0\rangle$ with adjoint $\langle 0|c_0(b_1 - c_1)$: this has norm

$$\langle 0|c_0(b_1 - c_1)(b_{-1} - c_{-1})|0\rangle = -\langle 0|c_0(\{b_1, c_{-1}\} + \{c_1, b_{-1}\})|0\rangle = -2\langle 0|c_0|0\rangle = -2 . \quad (30)$$

Relatedly, the state $(b_{-1} + c_{-1})|0\rangle$ has positive norm, while the states $b_{-1}|0\rangle$, $c_{-1}|0\rangle$, have zero norm. There is a plethora of negative norm states as we go to higher levels, built with oscillators as above.

3.2 The replica calculation and entanglement entropy

Revisiting the replica formulation of Calabrese, Cardy [32, 33], the calculation of entanglement entropy for a subsystem A consisting of a single interval is obtained as $S_{EE}^A = -\lim_{n \rightarrow 1} \partial_n \text{tr} \rho_A^n$ where $\text{tr} \rho_A^n$ is the path integral in the replica space consisting of n -copies of the original space, with adjacent copies appropriately sewn together, in the presence of a cut representing the interval. This partition function itself can be expressed in terms of a certain 2-point correlation function after a conformal transformation.

In more detail (see Appendix A), consider a 2-dim Euclidean CFT with central charge c , which we imagine to be a ghost CFT with negative central charge $c < 0$ of the sort above. Let us use complex coordinates $w = x + it_E$ and \bar{w} , where t_E is Euclidean time. From the point of view of regarding this as an intrinsically Euclidean theory, we are assuming translation invariance along one direction, which we take as Euclidean time: this allows us to decompose the Euclidean CFT into Euclidean time slices. Time evolution pertains to this Euclidean time and so does the entanglement entropy we are discussing for the subsystem in question, which is a spatial interval on a constant Euclidean time slice. So consider a subsystem A defined by a single interval stretched between $x = u$ and $x = v > u$ on a fixed time slice $t_E = \text{const}$. Under a conformal transformation $w \rightarrow z$, the energy-momentum tensor transforms as

$$T(w) = (\partial_w z)^2 T(z) + \frac{c}{12} \{z, w\} , \quad (31)$$

where $\{z, w\} = \frac{2\partial_w^3 z \partial_w z - 3(\partial_w^2 z)^2}{2(\partial_w z)^2}$ is the Schwarzian derivative. The replica w -space is transformed into the z -plane by the conformal transformation (96) given by $z = (\frac{w-u}{w-v})^{1/n}$ (see Appendix A). Assuming that on the z -plane there are no insertions of any nontrivial

operators gives

$$\langle T(z) \rangle_{\mathcal{C}} = 0 , \quad (32)$$

which is equivalent to taking the z -plane to represent the CFT ground state. Now taking expectation values (with \mathcal{R}_n the n -sheeted w -space), we have

$$\langle T(w) \rangle_{\mathcal{R}_n} = \frac{c}{12} \{z, w\} = \frac{c(1 - \frac{1}{n^2})}{24} \frac{(v - u)^2}{(w - u)^2(w - v)^2} = \frac{\langle T(w) \Phi_n(u) \Phi_{-n}(v) \rangle}{\langle \Phi_n(u) \Phi_{-n}(v) \rangle} . \quad (33)$$

The boundary conditions at u, v are equivalent to the insertion of twist operators $\Phi_n(u), \Phi_{-n}(v)$ at $w = u, v$ respectively. Comparing the last expression with the standard form for the $\langle T(z) O_1(z_1) O_2(z_2) \rangle$ 3-point function in CFT (see Appendix A) which is related to the 2-point function $\langle O_1(z_1) O_2(z_2) \rangle$, the conformal dimensions of the twist operators can be read off.

For the bc -ghost CFTs in question here, the formulation above must be refined further.

(1) Firstly, the condition (32) is equivalent to the statement that the z -plane represents the $SL(2)$ invariant vacuum $|0\rangle$ rather than the ghost ground state $|\downarrow\rangle$. It is the $SL(2)$ vacuum (13) which is defined by regularity of the energy-momentum tensor, and which naturally enters the condition (32) defining the z -plane after the conformal transformation (96), and where the resulting correlation functions can be studied. Equivalently the state-operator correspondence maps the identity operator to the $SL(2)$ vacuum through (13).

In general, the $SL(2)$ vacuum with $L_0 = 0$ is not the ghost ground state $|\downarrow\rangle$ which has $L_0|\downarrow\rangle = \frac{\lambda(1-\lambda)}{2}|\downarrow\rangle$, the L_0 -eigenvalue being negative for $\lambda > 1$. In the state $|\downarrow\rangle$, naively the condition (32) is not well-defined near $z = 0$, due to *e.g.* contributions such as $\frac{L_0}{z^2}|\downarrow\rangle$. Thus while the replica calculation in the $SL(2)$ vacuum appears to be formally valid, the interpretation is not entirely clear: the $SL(2)$ vacuum is perhaps best regarded as an excited state consisting of a partially filled Fermi sea (17), leaving the question of what happens in the ground state $|\downarrow\rangle$.

(2) Secondly, note that of course the condition $\langle T(z) \rangle_{\mathcal{C}} = 0$ is vacuous unless we use the normalizations of states given by (22). For instance, taking $\langle 0|T(z)|0\rangle$ instead gives a trivial condition since $\langle 0|0\rangle = 0$: the background charge inherent in the bc -ghost system is not cancelled without appropriate ghost zero mode insertions, equivalent to the normalizations (22). Thus (32) using the $SL(2)$ vacuum must be regarded as

$$\langle -Q| T(z) |0\rangle = 0 , \quad (34)$$

where Q is the background charge of the bc -ghost system, in accord with (22).

For the case $\lambda = 1$ corresponding to the bc -system with $c = -2$, the $SL(2)$ vacuum is the ghost ground state, and the state of lowest L_0 eigenvalue. Its adjoint is the state $\langle 0|c_0 = \langle \uparrow|$

which also has $L_0 = 0$. Although there is a plethora of negative norm states (norms defined via (22)), the L_0 eigenvalues are positive. Thus the replica formulation in the $SL(2)$ vacuum can be taken to represent the entanglement entropy in the ground state.

The n -sheeted replica theory is defined by boundary conditions

$$\begin{aligned} b_k(e^{2\pi i}(w-u)) &= b_{k+1}(w-u) , & c_k(e^{2\pi i}(w-u)) &= c_{k+1}(w-u) , \\ b_k(e^{2\pi i}(w-v)) &= b_{k-1}(w-v) , & c_k(e^{2\pi i}(w-v)) &= c_{k-1}(w-v) , \end{aligned} \quad k = 1, \dots, n, \quad (35)$$

so that $(b, c)_k \rightarrow (b, c)_{k+1}$ going around $w = u$ as $w - u \rightarrow e^{2\pi i}(w - u)$ and $(b, c)_k \rightarrow (b, c)_{k-1}$ under $w - v \rightarrow e^{2\pi i}(w - v)$, with the n -th sheet connecting back to the first. If one encircles both $w = u$ and $w = v$, the boundary conditions are trivial, *i.e.* far from the interval the replica space is essentially a single sheet. The fields and boundary conditions above can be diagonalized by defining

$$\tilde{b}_k = \frac{1}{n} \sum_{l=1}^n e^{2\pi i l k / n} b_l , \quad \tilde{c}_k = \frac{1}{n} \sum_{l=1}^n e^{-2\pi i l k / n} c_l . \quad (36)$$

This defines a twist around $w = u$

$$\tilde{b}_k(e^{2\pi i}(w-u)) = e^{-2\pi i k / n} \tilde{b}_k(w-u) , \quad \tilde{c}_k(e^{2\pi i}(w-u)) = e^{2\pi i k / n} \tilde{c}_k(w-u) , \quad (37)$$

and likewise an anti-twist around $w = v$. These are recognized as standard orbifold boundary conditions for conical singularities located at u, v . The conformal transformation argument for the stress tensor then leads to twist operators in the replica n -space with dimensions $\frac{c}{24}(1 - \frac{1}{n^2})$ with $c = -2$, as in our discussion in sec. 3: thus entanglement entropy in this CFT can be obtained as discussed there. Since these are Grassman variables, the treatment of the path integral is similar to that for fermions (discussed *e.g.* in [49, 50]).

To understand the replica boundary conditions in some more detail, we consider a \mathbb{Z}_N orbifold of the bc -theory with a twist by $e^{2\pi i k / N}$ around the origin defined by

$$b_t(e^{2\pi i} z) = e^{-2\pi i k / N} b_t(z) , \quad c_t(e^{2\pi i} z) = e^{2\pi i k / N} c_t(z) , \quad k = 1, \dots, N-1 , \quad (38)$$

and likewise for the anti-holomorphic sector. The mode expansions for the fields implementing these twist boundary conditions can be written as

$$b_t(z) = \sum_{m \in \mathbb{Z}} \frac{b_{m+k/N}}{z^{m+1+k/N}} , \quad c_t(z) = \sum_{m \in \mathbb{Z}} \frac{c_{m-k/N}}{z^{m-k/N}} . \quad (39)$$

The twist ground state $|0\rangle_{k/N}$ is annihilated by all operators with positive mode number, *i.e.* $b_{m+k/N}|0\rangle_{k/N} = 0$, $c_{m-k/N}|0\rangle_{k/N} = 0$, $m > 0$. The contour argument using the short distance behaviour as $z \rightarrow w$ away from the singularity leads to the anti-commutation relations

$$\{b_{m+k/N}, c_{n-k/N}\} = \delta_{m+n,0} . \quad (40)$$

Note that there is no zero mode in this orbifold twisted sector ($k \neq 0$), although the untwisted sector retains the b, c zero modes. The correlation function can be found using the mode expansions and the $b_{m+k/N}, c_{m-k/N}$ -operators as

$$\langle b_t(z) c_t(w) \rangle_{k/N} = \sum_{m,n} \frac{\langle 0 | b_{m+k/N} c_{n-k/N} | 0 \rangle_{k/N}}{z^{m+1+k/N} w^{n-k/N}} = \sum_{m=0}^{\infty} \frac{1}{z} \frac{w^{k/N}}{z^{k/N}} \left(\frac{w}{z} \right)^m = \left(\frac{z}{w} \right)^{-k/N} \frac{1}{z-w} . \quad (41)$$

The position dependence of this correlation function reflects the expected properties under going around the conical singularity as well as the short distance behaviour $\frac{1}{z-w}$ as $z \rightarrow w$ which is not affected by the boundary conditions. Likewise

$$\langle b_t(z) \partial c_t(w) \rangle_{k/N} = \sum_{m,n} \frac{\langle b_{m+k/N} c_{n-k/N} \rangle}{z^{m+1+k/N} w^{n+1-k/N}} = \frac{1}{z} \left(\frac{z}{w} \right)^{1-k/N} \frac{\frac{k}{N} z + (1 - \frac{k}{N}) w}{(z-w)^2} . \quad (42)$$

This is similar in form to the complex ($c = 2$) boson correlation function in [51], with a sign difference reflecting the short distance ghost structure.

In terms of the twist field $\sigma_{k/N}(z)$ creating the twist boundary conditions at location z , with appropriate OPEs that can be written as in *e.g.* [51], the twist ground state is $|0\rangle_{k/N} = \sigma_{k/N}(z)|0\rangle$. The boundary conditions (38) can then be defined in terms of the twist field $\sigma_{k/N}(z)$ as

$$b_t(e^{2\pi i} z) \sigma_{k/N}(0) = e^{-2\pi i k/N} b_t(z) \sigma_{k/N}(0) , \quad c_t(e^{2\pi i} z) \sigma_{k/N}(0) = e^{2\pi i k/N} c_t(z) \sigma_{k/N}(0) , \quad (43)$$

with $k = 1, \dots, N-1$. This is equivalent to the OPEs

$$b_t(z) \sigma_{k/N}(0) \sim z^{-k/N} \tau_{k/N} , \quad \partial c_t(z) \sigma_{k/N}(0) \sim z^{-1+k/N} \tau'_{k/N} , \quad (44)$$

etc, with $\tau_{k/N}, \tau'_{k/N}$ being excited twist fields. Corresponding to the twist field $\sigma_{k/N} \equiv \sigma_{k/N}^+$ here is the anti-twist field $\sigma_{k/N}^- \equiv \sigma_{1-k/N}^+$ defined via similar OPEs (or boundary conditions (43)) but with $\frac{k}{N} \rightarrow 1 - \frac{k}{N}$. Thus the bc correlation function (41) in this twist sector can be written as $\langle 0 | \sigma_{k/N}^- b_t(z) c_t(w) \sigma_{k/N}^+ | 0 \rangle$ with the twist and anti-twist operator inserted. The expectation value of the energy-momentum tensor in this twist ground state can be obtained by regularizing (point-splitting) as

$$\langle T(z) \rangle_{k/N} = \lim_{z \rightarrow w} \left(\langle - : b_t(z) \partial c_t(w) : \rangle + \frac{1}{(z-w)^2} \right) \quad (45)$$

Expanding (42) in $z-w$, the $\frac{1}{(z-w)^2}$ singularity cancels (effectively normal ordering $T(z)$) and we obtain

$$\langle T(z) \rangle_{k/N} = -\frac{1}{2} \frac{k}{N} \left(1 - \frac{k}{N} \right) \frac{1}{z^2} \equiv \frac{h_{\sigma_{k/N}}}{z^2} . \quad (46)$$

This gives the dimension of the twist field $\sigma_{k/N}(z)$, which we note is negative (this has also appeared previously in *e.g.* [37, 38]): this is a reflection of the short distance ghost behaviour and the negative central charge (and associated negative norm states). Likewise we can calculate the $U(1)$ charge of the twist field $\sigma_{k/N}^+$ as

$$\langle j(z) \rangle_{k/N} = \lim_{z \rightarrow w} \left(\langle - : b^t(z) c^t(w) : \rangle + \frac{1}{z - w} \right) = \frac{k/N}{z} , \quad (47)$$

so that the $U(1)$ charge of $\sigma_{k/N}^+$ is $\frac{k}{N}$ (see also [37, 38]). A similar calculation shows that the dimension of the anti-twist field $\sigma_{k/N}^- \equiv \sigma_{1-k/N}^+$ is $-\frac{1}{2} \frac{k}{N} (1 - \frac{k}{N})$ while its $U(1)$ charge is $1 - \frac{k}{N}$. Correlation functions of twist field operators also require the total $U(1)$ charge to cancel the background charge when calculated in the untwisted $SL(2)$ vacuum for being nonvanishing, *i.e.*

$$\langle \sigma_{\lambda_1}^+ \sigma_{\lambda_2}^+ \dots \rangle \equiv \langle 0 | \sigma_{\lambda_1}^+ \sigma_{1-\lambda_2}^- \dots | 0 \rangle \neq 0 \quad \Rightarrow \quad \sum_i \lambda_i = 1 , \quad (48)$$

reflecting the normalization (22). This is expected since for field insertions far from the region containing twist operator insertions, the OPEs resemble those in the untwisted theory so that correlation functions are nonvanishing only if the background charge is cancelled. In the bosonized formulation, we have $j(z) = i\partial\phi$ and $b(z) = e^{-\phi}$, $c(z) = e^{\phi}$, in the untwisted $c = -2$ theory. In the sector twisted by $\lambda = \frac{k}{N}$ the twist fields are $\sigma_\lambda = e^{i\lambda\phi} = \sigma_\lambda^+$. From this point of view it is clear in particular that a nonvanishing 2-point function is of the form $\langle 0 | \sigma_\lambda^+ \sigma_\lambda^- | 0 \rangle \equiv \langle 0 | \sigma_\lambda^+ \sigma_{1-\lambda}^- | 0 \rangle$, which automatically contains an unpaired c -field e^ϕ cancelling the background charge $Q = -1$.

Returning to the n -sheeted replica space defined by the boundary conditions (35), (36), (37), we note that the singularity at $w = u$ twists the b -field while that at $w = v$ anti-twists the b -field, and likewise for the c -field. In terms of the diagonalized fields \tilde{b}, \tilde{c} , in (36), the different sheets are decoupled: thus we can write the replica partition function as

$$tr \rho_A^n = \prod_{k=1}^{n-1} \langle \sigma_{k/N}^-(v) \sigma_{k/N}^+(u) \rangle_0 = (v - u)^{-4 \sum_{k=1}^{n-1} h_{\sigma_{k/N}}} = (v - u)^{\frac{1}{3}(n-1/n)} , \quad (49)$$

where the twist operator 2-point function is $\langle 0 | \sigma_{k/N}^-(v) \sigma_{k/N}^+(u) | 0 \rangle$ on sheet $-k$ for the $\lambda = 1$ bc -theory we are focussing on with $c = -2$ in the $SL(2)$ vacuum (equivalently the ghost ground state in this case). Since the neighbourhood of each singularity does not contain zero modes, one might be concerned about zero modes in the $n \rightarrow 1$ limit: the normalization (48) w.r.t. the background charge for twist field correlation functions (and more generally (22)) shows that the $n \rightarrow 1$ limit is smooth with regard to the correlation function being nonvanishing. This gives

$$S_A = -\lim_{n \rightarrow 1} \partial_n tr \rho_A^n = -\frac{2}{3} \log \frac{l}{\epsilon} , \quad (50)$$

where $l \equiv v - u$ and ϵ is the ultraviolet cutoff. This is of the standard form $\frac{c}{3} \log \frac{l}{\epsilon}$ with $c = -2$ and is thus negative.

It is noteworthy that negative central charge in the present context leads to twist operators with negative conformal dimension $h_{\sigma_{k/N}}$ as we have seen (46). This leads to a 2-point function of the form $\langle \sigma_{k/N}^-(v) \sigma_{k/N}^+(u) \rangle = |v - u|^{4(k/N)(1-k/N)}$. More generally the replica formulation gives $\langle \Phi_n \Phi_{-n} \rangle \sim |v - u|^{|c|(1-1/n^2)/6}$. Thus there is no short distance divergence here as $|v - u| \rightarrow 0$. However for large separations $|v - u| \rightarrow \infty$, the 2-point function diverges¹: this is an infrared divergence, and is reminiscent of long-distance instabilities in the replica theory. Note that in the limit $n \rightarrow 1$, the twist operators approach zero conformal dimension: this suggests that the replica CFT instability is possibly “marginal” at worst.

Finally (although the details are quite different) our discussion appears consistent with [52] who argue that the entanglement entropy has the form $c_{eff} \log \frac{l}{\epsilon}$. The effective central charge $c_{eff} = c - 24\Delta$ is often positive, with Δ the (negative) dimension of the operator with lowest conformal dimension. In the present bc -system with $c = -2$, the field of lowest conformal dimension is c with dimension $\Delta = h_c = 0$, so that $c_{eff} = c < 0$. It would be interesting to understand this better.

3.3 Matter + ghost systems

Let us now consider the bc -system (focussing on the $c = -2$ case) alongwith some other CFT with central charge $c_m > 1$: we have in mind a free CFT of say c_m free scalars. The total central charge of the full system is $c_m + c_{bc} = c_m - 2$. We assume the matter sector has a well-defined $SL(2)$ invariant ground state $|0_m\rangle$ with $L_0^m |0_m\rangle = 0$ so that the full theory has a $SL(2)$ invariant vacuum given by $|0_m\rangle \otimes |0_{bc}\rangle$, with $L_0^m + L_0^{bc} = 0$. Correlation functions in the ghost sector are defined with the normalizations (22) for $Q = -2$, while in the matter sector we assume $\langle 0_m | 0_m \rangle = 1$. Then from the previous arguments, the replica formulation for a single interval amounts to independent calculations in the matter and ghost sectors since the two sectors are free and decoupled from each other. In particular,

¹This has also been noted in [53] who study entanglement entropy in certain Q -state Potts models with $c < 0$, including numerical analysis. The Q -state Potts models, as statistical mechanical systems defined on an arbitrary lattice, exhibit a Q -valued permutation symmetry, the basic spin variable taking Q values (the Ising model being $Q = 2$), with spin-spin interactions. On a square lattice, with Q varying between 0 and 4, these exhibit critical points described by CFTs with central charge $c = 1 - \frac{6}{m(m+1)}$ and $\sqrt{Q} = 2 \cos \frac{\pi}{m+1}$. For $Q < 1$ (after appropriately defining the partition function), the CFT is non-unitary, with $c < 0$. In particular, for $Q = 2 - \sqrt{3}$, the CFT has central charge $c = -\frac{11}{14}$. [53] note that negative entanglement entropy suggests that the mixed state corresponding to the reduced density matrix is apparently more ordered than the ground state which has vanishing entropy. As a check, they note that the correlation function of the branch-point twist fields (with conformal dimension proportional to $c < 0$) grows with distance.

$tr \rho_A^n = \prod_k \langle \sigma_{k/N}^-(v) \sigma_{k/N}^+(u) \rangle_m \langle \sigma_{k/N}^-(v) \sigma_{k/N}^+(u) \rangle_{bc}$ with a product structure on the matter and ghost sectors. This leads to

$$S_A = \frac{c_m - 2}{3} \log \frac{l}{\epsilon} , \quad (51)$$

so that the entanglement entropy now effectively corresponds to that of a CFT with central charge $c_m - 2$. Perhaps this is not surprising since the ghost system acts to cancel central charge and corresponding conformal anomalies. In the present context, this suggests that the ghost system “cancels” entanglement, *i.e.* the ghosts effectively disentangle degrees of freedom of the original system. We mention that this again is formal: the physical interpretation of this entanglement cancellation is not clear at this point.

4 A ghost $\chi\bar{\chi}$ -CFT of anti-commuting scalars

The first order bc -ghost system with $c = -2$ discussed above can be thought of as a non-logarithmic sector of logarithmic conformal field theories studied in [42, 43, 44, 38, 39, 40, 41]. Consider a 2-dim Euclidean CFT with action $S_E = \int d^2\sigma (\partial_1\chi\partial_1\bar{\chi} + \partial_2\chi\partial_2\bar{\chi})$ defined on the Euclidean plane (σ^1, σ^2) , consisting of two anticommuting complex massless scalars $\chi, \bar{\chi}$, regarded as Grassmann variables. In the present case, we are motivated by [12] involving the 3-dim $Sp(N)$ Euclidean higher spin theory of anti-commuting scalars (see also [45, 46]): we consider the 2-dim ghost CFT here as a toy model, using standard CFT tools. Structurally this $\chi\bar{\chi}$ -CFT has some similarities with a complex commuting boson ϕ with action $S \sim \int d^2z (\partial\phi\bar{\partial}\bar{\phi} + \bar{\partial}\phi\partial\bar{\phi})$, for instance in its stress tensor $T(z) \sim -\partial\phi\partial\bar{\phi}$, operator product expansions (OPEs) and mode expansions etc, but with some crucial differences due to the Grassmann nature.

Defining $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$, gives the action and equations of motion

$$S = \int d^2z (\partial\chi\bar{\partial}\bar{\chi} + \bar{\partial}\chi\partial\bar{\chi}) , \quad \bar{\partial}\partial\chi = 0 , \quad \bar{\partial}\partial\bar{\chi} = 0 , \quad (52)$$

which then give the OPEs

$$\begin{aligned} \chi(z)\bar{\chi}(w) &\sim -\log|z-w|^2 , \\ i.e. \quad \partial_z\chi(z)\partial_w\bar{\chi}(w) &\sim -\frac{1}{(z-w)^2} , \quad \partial_w\bar{\chi}(w)\partial_z\chi(z) \sim \frac{1}{(z-w)^2} . \end{aligned} \quad (53)$$

This is similar to the commuting boson case, except for the relative minus sign in the second line due to the anti-commuting nature of $\chi, \bar{\chi}$. The presence of the two derivatives makes the $\partial\chi\partial\bar{\chi}$ OPE differ from that for $\partial\bar{\chi}\partial\chi$ by a minus sign, consistent with the short-distance singularity being $\frac{-1}{(z-w)^2}$ rather than *e.g.* $\frac{1}{z-w}$ as in the bc -ghost CFT. From the

action $S \sim \int d^2\sigma \sqrt{g} g^{ij} \partial_i \chi \partial_j \bar{\chi}$, the energy-momentum tensor is

$$T_{ij} \sim -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{ij}} \sim -\partial_i \chi \partial_j \bar{\chi} + \frac{1}{2} g_{ij} L \quad \Rightarrow \quad T_{zz} \sim -\partial \chi \partial \bar{\chi}, \quad T_{\bar{z}\bar{z}} \sim -\bar{\partial} \chi \bar{\partial} \bar{\chi}, \quad (54)$$

where we evaluate $T_{zz} = \frac{1}{4}(T_{11} - T_{22} - 2iT_{12})$ using complex coordinates, and so on (or via Noether's theorem). It can be checked that $T_{z\bar{z}} = 0$ which reflects conformal invariance of the theory. The TT OPE has the standard form with $c = -2$: we have

$$T(z)T(w) = : \partial \chi \partial \bar{\chi}(z) : : \partial \chi \partial \bar{\chi}(w) : = -\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{(z-w)} \partial T(w). \quad (55)$$

The $c = -2$ central term with the minus sign arises from moving $\bar{\chi}$ through the other fields, since $\chi, \bar{\chi}$ anticommute. The OPE $T(z)\partial\chi(0) \sim \frac{1}{z^2}\partial\chi(0) + \frac{1}{z}\partial^2\chi(0)$ of $T(z)$ with $\partial\chi$ (and likewise $\partial\bar{\chi}$) is similar to that of a complex commuting $c = 2$ scalar and show that $\partial\chi$ and $\partial\bar{\chi}$ have conformal dimension one. From the equations of motion $\bar{\partial}\partial\chi = 0$, $\partial\bar{\partial}\bar{\chi} = 0$, we see that $\partial\chi$ and $\partial\bar{\chi}$ are holomorphic giving the mode expansions

$$\partial\chi = -i \sum_{m=-\infty}^{\infty} \frac{\chi_m}{z^{m+1}}, \quad \partial\bar{\chi} = -i \sum_{m=-\infty}^{\infty} \frac{\bar{\chi}_m}{z^{m+1}}. \quad (56)$$

Inverting we have $\chi_m = \oint \frac{dz}{2\pi} z^m \partial\chi$, $\bar{\chi}_m = \oint \frac{dz}{2\pi} z^m \partial\bar{\chi}$. The OPEs (53) and likewise $\partial_z \chi \partial_w \chi \sim \text{reg}$ and $\partial_z \bar{\chi} \partial_w \bar{\chi} \sim \text{reg}$ can be used via the usual contour arguments, *e.g.* $\{\chi_m, \bar{\chi}_n\} = \oint_{C_2} \frac{dw}{2\pi} \text{Res}_{z \rightarrow w} z^m w^n \partial_z \chi \partial_w \bar{\chi}$. These give the anti-commutation relations for the oscillators,

$$\{\chi_m, \bar{\chi}_n\} = m \delta_{m+n,0}, \quad \{\chi_m, \chi_n\} = 0, \quad \{\bar{\chi}_m, \bar{\chi}_n\} = 0. \quad (57)$$

In particular this means $\{\chi_0, \bar{\chi}_0\} = 0$ due to the m factor. From the mode expansions (56), we obtain (for the holomorphic parts)

$$\chi(z) = \xi_0 - i\chi_0 \log z - i \sum_{m \neq 0} \frac{\chi_m}{m z^m}, \quad \bar{\chi}(z) = \bar{\xi}_0 - i\bar{\chi}_0 \log z - i \sum_{m \neq 0} \frac{\bar{\chi}_m}{m z^m}, \quad (58)$$

with $\xi_0, \bar{\xi}_0$ the constant parts or the zero modes of the $\chi, \bar{\chi}$ -fields. From this and the OPEs (53), we find the additional anti-commutation relations

$$\{\xi_0, \bar{\chi}_0\} = i, \quad \{\bar{\xi}_0, \chi_0\} = -i. \quad (59)$$

The presence of the zero modes $\xi_0, \bar{\xi}_0$, leads to similarities with the bc -ghost systems earlier, but with additional features here. In particular, there are logarithmic operators in this theory, *e.g.*

$$\tilde{\mathbb{I}} = : \chi \bar{\chi} :, \quad T(z)\tilde{\mathbb{I}}(w) = \frac{\mathbb{I}}{(z-w)^2} + \frac{1}{z-w} \partial \tilde{\mathbb{I}} + \dots, \quad (60)$$

with \mathbb{I} the identity operator, and $\tilde{\mathbb{I}}$ its logarithmic partner. Thus $[L_0, \tilde{\mathbb{I}}] = \mathbb{I}$, so that L_0 cannot be completely diagonalized, giving a 2-component “Jordan cell” with L_0 eigenvalue zero. These features give rise to logarithms in some correlation functions: the conditions on correlation functions following from conformal invariance lead to coupled differential equations whose solutions involve power-laws z^α , as well as $z^\alpha \log z$ (see *e.g.* (69) later), as discussed in [42] and also [43, 44, 38, 39, 40, 41].

Using the expansion $T(z) = \sum_m \frac{L_m}{z^{m+2}}$ and the OPEs of $T(z)$ with $\partial\chi$, $\partial\bar{\chi}$, it can be seen that the subsector comprising excited states built over the ground state $|0\rangle$ by oscillators, *e.g.* $\prod \chi_{-n} \bar{\chi}_{-m} |0\rangle$, have positive $L_0 \sim \sum_n (-\chi_{-n} \bar{\chi}_n + \bar{\chi}_{-n} \chi_n)$ eigenvalue in this subspace (with $L_m \sim -\sum_n \chi_n \bar{\chi}_{m-n}$). These are fermionic excitations since $\chi_k^2, \bar{\chi}_k^2 = 0$. It is useful to recall that for fermionic oscillator operators satisfying $\{a, a^\dagger\} = 1$, a Hamiltonian $H = a^\dagger a$ gives $[H, a] = -a$, $[H, a^\dagger] = a^\dagger$ so that one can define a ground state as the lowest weight state with $a|vac\rangle = 0$. However for oscillators with $\{a, a^\dagger\} = -1$, we have $[a^\dagger a, a] = a$, $[a^\dagger a, a^\dagger] = -a^\dagger$. However defining $H = -a^\dagger a$ gives $[H, a] = -a$, $[H, a^\dagger] = -a^\dagger$ allowing for excited states built over the ground state regarded as a lowest weight state. In the present case, the subsector built over the vacuum $|0\rangle$ has a representation for L_0 which gives $|0\rangle$ as a lowest weight state, with $[L_0, \chi_{-n}] = n\chi_{-n}$, $[L_0, \bar{\chi}_{-n}] = n\bar{\chi}_{-n}$, $[L_0, \chi_n] = -n\chi_n$, $[L_0, \bar{\chi}_n] = -n\bar{\chi}_n$, ($n > 0$), consistent with $\chi_{-n}, \bar{\chi}_{-n}$, $n > 0$ being creation operators, and $\chi_n, \bar{\chi}_n$, $n > 0$ being annihilation operators.

Defining the “logarithmic” state

$$|\xi_0\rangle \equiv \xi_0 \bar{\xi}_0 |0\rangle, \quad (61)$$

we see that due to the Grassmann integration over the zero modes $\xi_0, \bar{\xi}_0$, the vacuum of this theory $|0\rangle$ satisfies

$$\langle 0|0\rangle = 0, \quad \langle \xi_0|0\rangle = \langle 0|\xi_0 \bar{\xi}_0|0\rangle = 1, \quad (62)$$

where inserting the zero modes serves to give a nonzero answer in the Grassmann integration. From the path integral point of view, we have $\langle 0|0\rangle = \int D\chi D\bar{\chi} e^{-S} = 0$ whereas $\langle \xi_0|0\rangle = \int D\chi D\bar{\chi} \chi \bar{\chi} e^{-S} = 1$. Inserting a single logarithmic operator $\tilde{\mathbb{I}}$ serves to ensure nonvanishing results for correlation functions of derivative operators built out of $\partial\chi, \partial\bar{\chi}$ (56), which do not contain the zero modes: this is a minimal way to restrict to a nonlogarithmic subsector of this theory. In particular,

$$\langle \xi_0| \partial\chi(z) \partial\bar{\chi}(w) |0\rangle \equiv \langle 0|\xi_0 \bar{\xi}_0 \partial\chi(z) \partial\bar{\chi}(w) |0\rangle = -\frac{1}{(z-w)^2}. \quad (63)$$

Thus $|\xi\rangle = \xi_0 \bar{\xi}_0 |0\rangle$ can be regarded as the out-state for the in-state being the vacuum $|0\rangle$, analogous to (22) for the bc -ghost system. With the norms of states defined in this manner, correlation functions of derivative operators (which do not contain the zero modes) have

form similar to those of the commuting boson, with no additional logarithmic structure. We can use the mode expansions (56) to obtain the 2-point function as $\langle \xi_0 | (-\sum_{n>0} n \frac{w^{n-1}}{z^{n+1}}) | 0 \rangle = -\langle \xi_0 | 0 \rangle \frac{1}{z^2} \sum_{n=1}^{\infty} n \frac{w^{n-1}}{z^{n+1}} = -\frac{1}{(z-w)^2}$ with the normalization (62) for the “inner product”.

Likewise we see negative norm states using (62) and defining hermiticity relations in the following way: by analogy with the complex boson, it is consistent to take the hermitian conjugate field to χ as $\chi^\dagger = \bar{\chi}$ which gives $T(z)^\dagger = T(z)$. In the Euclidean theory, the hermitian conjugate can be defined as $\partial\chi(z)^\dagger \equiv \partial\bar{\chi}(\bar{z})$ where $\phi(z)^\dagger \equiv \phi^\dagger(\frac{1}{\bar{z}})\frac{1}{\bar{z}^{2h}}$ with h the conformal dimension of ϕ . This gives $-i\sum_n \frac{\bar{\chi}_n}{z^{n+1}} = \sum_n i \frac{\chi_n^\dagger}{\bar{z}^{n+1}} = \sum_n i \frac{\chi_{-n}^\dagger}{z^{n+1}}$ as usual for holomorphic field mode expansions, suggesting the hermitian conjugate operators $\chi_n^\dagger = -\bar{\chi}_{-n}$, $\chi_{-n}^\dagger = -\bar{\chi}_n$ ($n > 0$) and $\chi_0^\dagger = -\bar{\chi}_0$. This gives $L_n^\dagger = L_{-n}$ which is equivalent to $T(z)^\dagger = T(z)$. This is consistent with the oscillator algebra (57), $\{\chi_m, \bar{\chi}_{-m}\} = m$. Then with the norms defined as (62), we see that states of the form $\bar{\chi}_{-m}|0\rangle$, $m > 0$ have negative norm, since $\langle \xi_0 | (-\chi_m)\bar{\chi}_{-m} | 0 \rangle = -m < 0$. States of the form $\chi_{-m}|0\rangle$ have positive norm: more generally, states like $\prod \chi_{-m_i} \bar{\chi}_{-n_j} | 0 \rangle$ have negative norm for $\sum_i m_i < \sum_j n_j$ (but positive L_0 eigenvalue).

Entanglement entropy: We can formulate entanglement entropy through the replica as in the $c = -2$ bc -ghost system (sec. 3.2): a possible concern is the presence of logarithmic operators, as discussed in [42, 43, 44, 38, 39, 40, 41], which can lead to further modifications to the logarithmic behaviour in the entanglement entropy. Before discussing this, we note that the n -sheeted replica theory is defined by boundary conditions

$$\begin{aligned} \chi_k(e^{2\pi i}(w-u)) &= \chi_{k+1}(w-u) , & \bar{\chi}_k(e^{2\pi i}(w-u)) &= \bar{\chi}_{k+1}(w-u) , & k &= 1, \dots, n-1, \\ \chi_k(e^{2\pi i}(w-v)) &= \chi_{k-1}(w-v) , & \bar{\chi}_k(e^{2\pi i}(w-v)) &= \bar{\chi}_{k-1}(w-v) , & & \end{aligned} \quad (64)$$

with the n -th sheet connecting back to the first. Encircling both $w = u$ and $w = v$, gives trivial boundary conditions, *i.e.* far from the interval the replica space is essentially a single sheet. The fields and boundary conditions above can be diagonalized by defining $\tilde{\chi}_k = \frac{1}{n} \sum_{l=1}^n e^{-2\pi i l k / n} \chi_k$, $\tilde{\bar{\chi}}_k = \frac{1}{n} \sum_{l=1}^n e^{2\pi i l k / n} \bar{\chi}_k$. This defines a twist $\tilde{\chi}_k(e^{2\pi i}(w-u)) = e^{-2\pi i k / n} \tilde{\chi}_k(w-u)$, $\tilde{\bar{\chi}}_k(e^{2\pi i}(w-u)) = e^{2\pi i k / n} \tilde{\bar{\chi}}_k(w-u)$, around $w = u$, and likewise an anti-twist around $w = v$. These are standard orbifold boundary conditions for conical singularities located at u, v . To illustrate this, consider a \mathbb{Z}_N orbifold of the $\chi\bar{\chi}$ -theory with a twist by $e^{2\pi i k / N}$ around the origin

$$\chi(e^{2\pi i} z) = e^{2\pi i k / N} \chi(z) , \quad \bar{\chi}(e^{2\pi i} z) = e^{-2\pi i k / N} \bar{\chi}(z) , \quad k = 1, \dots, N-1 , \quad (65)$$

and likewise for the anti-holomorphic sector. The mode expansions for the fields implementing these twist boundary conditions and the corresponding anti-commutation relations

are

$$\partial\chi = \sum_m \frac{\chi_{m-k/N}}{z^{m+1-k/N}} , \quad \partial\bar{\chi} = \sum_m \frac{\bar{\chi}_{m+k/N}}{z^{m+1+k/N}} , \quad \{\chi_{m-k/N}, \bar{\chi}_{m+k/N}\} = \left(m - \frac{k}{N}\right) \delta_{m+n,0} . \quad (66)$$

The twist ground state $|0\rangle_{k/N}$ is annihilated by all operators with positive mode number, *i.e.* $\chi_{m-k/N}|0\rangle_{k/N} = 0$, and $\bar{\chi}_{m+k/N}|0\rangle_{k/N} = 0$, $m \geq 0$, and the anti-commutation relations are obtained by the contour argument using the short distance behaviour as $z \rightarrow w$ away from the singularity. The twist ground state can be thought of as $|0\rangle_{k/N} = \sigma_{k/N}(z)|0\rangle$, with $\sigma_{k/N}(z)$ the twist field creating the twist boundary conditions at location z . The correlation function can be found using these mode expansions and the $\chi_{m-k/N}, \bar{\chi}_{m+k/N}$ -operators as

$$\langle \partial\chi(z) \partial\bar{\chi}(w) \rangle_{k/N} = \sum_{m,n} \frac{\langle \chi_{m-k/N} \bar{\chi}_{n+k/N} \rangle}{z^{m+1-k/N} w^{n+1+k/N}} = \frac{1}{z} \left(\frac{z}{w}\right)^{k/N} \frac{(1 - \frac{k}{N})z + \frac{k}{N}w}{(z-w)^2} . \quad (67)$$

This is similar in form to the ($c = 2$) complex boson correlation function in [51], except for a minus sign as expected for these anti-commuting $c = -2$ scalars: it is also similar in form to (42) in the bc -ghost system discussed previously. The expectation value of the energy-momentum tensor in this twist ground state can be obtained by regularizing as $\langle T(z) \rangle_{k/N} = \lim_{z \rightarrow w} (\langle -\partial\chi(z) \partial\bar{\chi}(w) \rangle + \frac{1}{(z-w)^2})$, and we obtain

$$\langle T(z) \rangle_{k/N} = -\frac{1}{2} \frac{k}{N} \left(1 - \frac{k}{N}\right) \frac{1}{z^2} \equiv \frac{h_{\sigma_{k/N}}}{z^2} . \quad (68)$$

This gives the dimension of the twist field $\sigma_{k/N}(z)$. Not surprisingly this is the same as that for the complex boson, except for the minus sign.

As mentioned earlier, the presence of the logarithmic operators in these logarithmic conformal field theories (equivalently the fact that L_0 is not completely diagonalizable) leads to logarithms in correlation functions, rather than simple power laws alone, as discussed in [42, 43, 44, 38, 39, 40, 41]. In particular inserting multiple logarithmic operators leads to logarithmic behaviour in correlation functions, with *e.g.* $\langle \tilde{\mathbb{I}}(z) \tilde{\mathbb{I}}(w) \rangle \sim -\log(z-w)$. The OPEs of twist field operators in these theories also contain logarithms, *e.g.*

$$\mu(z)\mu(0) \sim z^{1/4} (\mathbb{I} \log z + \tilde{\mathbb{I}}) \quad (69)$$

with a Z_2 twist, where the twist fields μ have conformal dimension $-\frac{1}{8}$. More generally, for orbifolds with twists of the form (65), we have [39]

$$\sigma_{k/N}^-(z) \sigma_{k/N}^+(w) \sim z^{-2h_{\sigma_{k/N}}} (\mathbb{I} \log(z-w) + \tilde{\mathbb{I}}) . \quad (70)$$

A 2-point correlation function in the vacuum $|0\rangle$ takes the form [39]

$$\langle 0 | \sigma_{k/N}^-(z, \bar{z}) \sigma_{k/N}^+(w, \bar{w}) | 0 \rangle \sim |z-w|^{-4h_{\sigma_{k/N}}} = |z-w|^{2(k/N)(1-k/N)} , \quad (71)$$

where only the term with the single logarithmic operator $\tilde{\mathbb{I}}$ in the OPE contributes: this is in accord with the norm (62). Thus the 2-point function has power law behaviour, with no logarithm. The correlation function $\langle \sigma_{k/N}^-(z_1) \sigma_{k/N}^+(z_2) \tilde{\mathbb{I}}(z_3) \rangle$ exhibits logarithmic behaviour, with a form $|z_{12}|^{2(k/N)(1-k/N)} (\alpha + \beta \log |\frac{z_{13}z_{23}}{z_{12}}|^2)$ with appropriate coefficients α, β . More generally, higher point correlation functions of twist field operators (arising in *e.g.* entanglement entropy for multiple disjoint intervals, or mutual information) are expected to exhibit the logarithms characteristic of these log-CFTs.

In terms of the diagonalized fields $\tilde{\chi}$, we see that the different sheets are decoupled and thus we can write the replica partition function as a product of 2-point functions of twist operators,

$$\text{tr} \rho_A^n = \prod_{k=0}^{n-1} \langle \sigma_{k/N}^-(v) \sigma_{k/N}^+(u) \rangle = (v-u)^{-4 \sum_{k=0}^{n-1} h_{\sigma_{k/N}}} = (v-u)^{\frac{1}{3}(n-1/n)}. \quad (72)$$

This gives

$$S_A = -\lim_{n \rightarrow 1} \partial_n \text{tr} \rho_A^n = -\frac{2}{3} \log \frac{l}{\epsilon}, \quad (73)$$

where $l \equiv v-u$ and ϵ is the ultraviolet cutoff. This is of the standard form $\frac{c}{3} \log \frac{l}{\epsilon}$ with $c = -2$ and is negative. This is similar to the bc -ghost system which is in a sense a non-logarithmic subsector of the logarithmic CFT here.

5 A toy model of “ghost-spins”

To abstract away from the specific technical issues of the bc -ghost system, let us consider a very simple toy model of “ghost-spins” below, which mimics some of the key features. Firstly, for ordinary spin variables with a 2-state Hilbert space consisting of $\{\uparrow, \downarrow\}$, we take the usual positive definite norms in the Hilbert space $\langle \uparrow | \uparrow \rangle = \langle \downarrow | \downarrow \rangle = 1$ and $\langle \uparrow | \downarrow \rangle = \langle \downarrow | \uparrow \rangle = 0$. Then a generic state is $|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle$, with adjoint $\langle \psi| = c_1^* \langle \uparrow| + c_2^* \langle \downarrow|$ and norm $\langle \psi | \psi \rangle = |c_1|^2 + |c_2|^2$, which is positive definite. Thus we can normalize states as $\langle \psi | \psi \rangle = 1$ and pick a representative ray with unit norm (equivalent to calculating expectation values of operators as $\langle O \rangle = \frac{\langle \psi | O | \psi \rangle}{\langle \psi | \psi \rangle} = \langle \psi | O | \psi \rangle$). The reduced density matrix obtained by tracing out the second spin is

$$\rho_A = \text{tr}_B |\psi\rangle \langle \psi| = \sum_i \langle i_B | \psi \rangle \langle \psi | i_B \rangle = \langle \uparrow_B | \psi \rangle \langle \psi | \uparrow_B \rangle + \langle \downarrow_B | \psi \rangle \langle \psi | \downarrow_B \rangle. \quad (74)$$

The familiar discussions in 2-spin systems of entanglement entropy via the reduced density matrix are recovered as follows. States of the system such as $|\psi\rangle = c_1 |\uparrow\uparrow\rangle + c_2 |\downarrow\downarrow\rangle$ can be normalized as $\langle \psi | \psi \rangle = 1 = |c_1|^2 + |c_2|^2$ which is positive definite, and ensure that $|c_1|, |c_2| \leq 1$.

With these norms, the reduced density matrix (74) becomes $\rho_A = |c_1|^2 |\uparrow\rangle\langle\uparrow| + |c_2|^2 |\downarrow\rangle\langle\downarrow|$. Note that the reduced density matrix is automatically normalized as $\text{tr}\rho_A = 1$ once the state $|\psi\rangle$ is normalized. Thus the entanglement entropy given as the von Neumann entropy of ρ_A is $S_A = -\text{tr}\rho_A \log \rho_A = -\sum_i \rho_A(i) \log \rho_A(i)$ is positive definite since each eigenvalue $\rho_A(i) < 1$ makes the $-\log \rho_A(i) > 0$.

We define a single “ghost-spin” by a similar 2-state Hilbert space consisting of $\{\uparrow, \downarrow\}$, but defining the “norms” as

$$\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0, \quad \langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1. \quad (75)$$

This is akin to the normalizations (22) in the bc -ghost system (see also [34], Appendix, vol. 1 where this inner product appears). Now a generic state

$$|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle \quad \Rightarrow \quad \langle\psi|\psi\rangle = c_1 c_2^* + c_2 c_1^*, \quad (76)$$

which is not positive definite: for instance the state $|\uparrow\rangle - |\downarrow\rangle$ has norm -2 . It is then convenient to change basis to

$$|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle), \quad \langle++\rangle = 1, \quad \langle--\rangle = -1, \quad \langle+-\rangle = \langle-+\rangle = 0. \quad (77)$$

A generic state with nonzero norm can be normalized to norm $+1$ or -1 . Then a negative norm state can be written as

$$|\psi\rangle = \frac{1}{\sqrt{|c_2|^2 - |c_1|^2}}(c_1|+\rangle + c_2|-\rangle), \quad \langle\psi|\psi\rangle = -1 \quad (|c_2|^2 > |c_1|^2). \quad (78)$$

For every state (or ray) $|\psi\rangle$ with norm -1 , there is a corresponding state (or ray) $|\psi^\perp\rangle$ with norm $+1$ which is orthogonal to $|\psi\rangle$,

$$|\psi^\perp\rangle = \frac{1}{\sqrt{|c_2|^2 - |c_1|^2}}(c_2^*|+\rangle + c_1^*|-\rangle), \quad \langle\psi^\perp|\psi^\perp\rangle = 1, \quad \langle\psi^\perp|\psi\rangle = 0. \quad (79)$$

There are also zero norm states given by

$$|\psi\rangle = c_1|+\rangle + c_2|-\rangle, \quad \langle\psi|\psi\rangle = 0, \quad |c_2|^2 = |c_1|^2, \quad (80)$$

which do not admit any canonical normalization, simple examples being $|\uparrow\rangle, |\downarrow\rangle$.

Now considering the two ghost-spin system, basis states are

$$|s_A s_B\rangle \equiv |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \equiv |++\rangle, |+-\rangle, |-+\rangle, |--\rangle. \quad (81)$$

The $|\pm\pm\rangle$ basis is more transparent for our purposes. Since the inner product or metric on this space of states is not positive definite, we need to be careful in defining the various contractions arising in the norms and partial traces. We define the states and norms as

$$|\psi\rangle = \sum \psi_{ij} |ij\rangle, \quad \langle\psi|\psi\rangle \equiv g^{ik} g^{jl} \psi_{ij} \psi_{kl}^* = g^{ii} g^{jj} |\psi_{ij}|^2, \quad (82)$$

where repeated indices as usual are summed over: the last expression pertains to the $|\pm\rangle$ basis where the metric is diagonal, with $g^{++} = 1$, $g^{--} = -1$. Thus a simple example of a positive norm state is $|\psi\rangle = \psi_{++}|++\rangle + \psi_{--}|--\rangle$ with $\langle\psi|\psi\rangle = |\psi_{++}|^2 + |\psi_{--}|^2 > 0$, while $|\psi\rangle = \psi_{+-}|+-\rangle + \psi_{-+}| - + \rangle$ is a negative norm state with $\langle\psi|\psi\rangle = -(|\psi_{+-}|^2 + |\psi_{-+}|^2) < 0$. Then

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{c^2}}(c_1|++\rangle + c_2|+-\rangle + c_3| - + \rangle + c_4|--\rangle) \\ \Rightarrow \quad \langle\psi|\psi\rangle &= \frac{1}{c^2}(|c_1|^2 + |c_4|^2 - |c_2|^2 - |c_3|^2), \end{aligned} \quad (83)$$

is a generic state, so that normalized positive/negative norm states have

$$c^2 = \pm(|c_1|^2 + |c_4|^2 - |c_2|^2 - |c_3|^2) > 0 \quad \Rightarrow \quad \langle\psi|\psi\rangle = \pm 1, \quad (84)$$

with norm ± 1 . This translates to corresponding conditions on the coefficients c_i .

With the density matrix $\rho = |\psi\rangle\langle\psi| = \sum \psi_{ij}\psi_{kl}^*|ij\rangle\langle kl|$, the reduced density matrix obtained by a partial trace over one spin can again be defined via a partial contraction as

$$\rho_A = tr_B \rho \equiv (\rho_A)_{ik}|i\rangle\langle k|, \quad (\rho_A)_{ik} = g^{jl}\psi_{ij}\psi_{kl}^* = g^{jj}\psi_{ij}\psi_{kj}^*. \quad (85)$$

This gives

$$\begin{aligned} (\rho_A)_{++} &= |\psi_{++}|^2 - |\psi_{+-}|^2, & (\rho_A)_{+-} &= \psi_{++}\psi_{-+}^* - \psi_{+-}\psi_{--}^*, \\ (\rho_A)_{-+} &= \psi_{-+}\psi_{++}^* - \psi_{--}\psi_{+-}^*, & (\rho_A)_{--} &= |\psi_{-+}|^2 - |\psi_{--}|^2, \end{aligned} \quad (86)$$

For the state (83), (84), this is

$$\begin{aligned} \rho_A &= \frac{1}{c^2} \left[(|c_1|^2 - |c_2|^2)|+\rangle\langle +| + (c_1c_3^* - c_2c_4^*)|+\rangle\langle -| \right. \\ &\quad \left. + (c_3c_1^* - c_4c_2^*)|-\rangle\langle +| + (|c_3|^2 - |c_4|^2)|-\rangle\langle -| \right]. \end{aligned} \quad (87)$$

Then $tr \rho_A = g^{ik}(\rho_A)_{ik} = (\rho_A)_{++} - (\rho_A)_{--}$. Thus the reduced density matrix is normalized to have $tr \rho_A = tr \rho = \pm 1$ depending on whether the state (83), (84), is positive or negative norm. Also, ρ_A has some eigenvalues negative.

The entanglement entropy calculated as the von Neumann entropy of the reduced density matrix is

$$S_A = -g^{ij}(\rho_A \log \rho_A)_{ij} = -g^{++}(\rho_A \log \rho_A)_{++} - g^{--}(\rho_A \log \rho_A)_{--} \quad (88)$$

where the last expression pertains to the $|\pm\rangle$ basis with $g^{\pm\pm} = \pm 1$. This requires defining $\log \rho_A$ as an operator²: we define this as

$$(\log \rho_A)_{ik} = (\log(1 + \rho_A - 1))_{ik} = 1_{ik} + (\rho_A - 1)_{ik} + (\rho_A - 1)_{ij}g^{jl}(\rho_A - 1)_{lk} + \dots \quad (89)$$

²I thank D. Jatkar, A. Maharana and A. Sen for useful discussions here.

or equivalently as the solution to $(\rho_A)_{ik} = (e^{\log \rho_A})_{ik} = 1_{ik} + (\log \rho_A)_{ik} + (\log \rho_A)_{ij} g^{jl} (\log \rho_A)_{lk} + \dots$. The signs in the contractions in $\log \rho_A$ are perhaps more easily dealt with if we use the mixed-index reduced density matrix $(\rho_A)^i_k$.

To illustrate this, let us for simplicity consider a simple family of states where the reduced density matrix is diagonal, by restricting to $c_3^* = \frac{c_2 c_4^*}{c_1}$. In this case, $\log \rho_A$ is also diagonal and can be calculated easily. From (87) for the state (83), (84), this gives

$$c_3^* = \frac{c_2 c_4^*}{c_1} \quad \Rightarrow \quad c^2 = \pm (|c_1|^2 - |c_2|^2) \left(1 + \frac{|c_4|^2}{|c_1|^2} \right) > 0 ,$$

$$\rho_A = \pm \left[\frac{|c_1|^2}{|c_1|^2 + |c_4|^2} |+\rangle\langle +| - \frac{|c_4|^2}{|c_1|^2 + |c_4|^2} |-\rangle\langle -| \right] , \quad (90)$$

where the \pm refer to positive and negative norm states respectively. The location of the negative eigenvalue is different for positive and negative norm states, leading to different results for the von Neumann entropy. For negative norm states, we see that $c^2 > 0$ implies $|c_2|^2 > |c_1|^2$ and $|c_3|^2 > |c_4|^2$, so that $(\rho_A)_{++} < 0$, $(\rho_A)_{--} > 0$. Then the mixed-index reduced density matrix components $(\rho_A)^i_k = g^{ij}(\rho_A)_{jk}$ are

$$(\rho_A)_+^+ = \pm x , \quad (\rho_A)_-^- = \pm(1 - x) , \quad x = \frac{|c_1|^2}{|c_1|^2 + |c_4|^2} , \quad 0 < x < 1 . \quad (91)$$

Thus we see that $\text{tr} \rho_A = (\rho_A)_+^+ + (\rho_A)_-^- = \pm 1$ manifestly. Now we obtain

$$(\log \rho_A)_+^+ = \log(\pm x) , \quad (\log \rho_A)_-^- = \log(\pm(1 - x)) , \quad (92)$$

the \pm referring again to positive/negative norm states respectively. Thus the entanglement entropy (88) becomes

$$S_A = -(\rho_A)_+^+ (\log \rho_A)_+^+ - (\rho_A)_-^- (\log \rho_A)_-^- \quad (93)$$

For positive norm states, we obtain

$$\langle \psi | \psi \rangle > 0 : \quad S_A = -x \log x - (1 - x) \log(1 - x) > 0 , \quad (94)$$

which is manifestly positive since $x < 1$, just as for the familiar entanglement entropy in an ordinary 2-spin system. For negative norm states however, we have

$$\langle \psi | \psi \rangle < 0 : \quad S_A = x \log(-x) + (1 - x) \log(-(1 - x)) = x \log x + (1 - x) \log(1 - x) + i\pi . \quad (95)$$

We note that the imaginary part (using $\log(-1) = i\pi$) is independent of x , *i.e.* the same for all such negative norm states. The real part of entanglement entropy is negative since $x < 1$ and the logarithms are negative: apart from the minus sign, it is the same as S_A for the positive

norm states. This real part is minimized when $x = \frac{1}{2}$ (this value corresponds to maximal entanglement for positive norm states): this “minimal” entanglement is $S_A = -\log 2 + i\pi$.

The above discussion can also be phrased in terms of the $|\uparrow\rangle, |\downarrow\rangle$ basis although we have found it convenient to use the $|\pm\rangle$ basis. It is worth noting that while (75) mimics the ghost norms (22), there is no obvious analog of the background charge here: in particular tracing over spin_A instead of spin_B is equivalent, so that entanglement entropy for the subsystem is the same as that for the complement.

It would appear that this discussion can be generalized to an arbitrary lattice L containing ghost-spins $|\uparrow_i\rangle, |\downarrow_i\rangle$ at each lattice site $i \in L$. Using the $|\pm_i\rangle$ basis, states with negative norm can be constructed as above, and a subsystem can presumably be defined as a connected spatial subregion in the lattice. It would be interesting to explore entanglement entropy in these ghost-spin systems more completely, as well as coupling to ordinary spin systems.

6 Discussion

We have studied entanglement entropy in some Euclidean non-unitary CFTs with a view to gaining more insight into the areas of the complex extremal surfaces on a constant Euclidean time slice in de Sitter space (Poincare slicing) studied in [19, 20]. In particular for dS_4 , the areas are negative reflecting the negative central charge $-\frac{R_{dS}^2}{G_4}$. With a view to finding toy models exhibiting negative entanglement entropy, we have studied 2-dim ghost CFTs with negative central charge, revisiting the replica formulation of entanglement entropy for a single interval. In particular in the bc -ghost system with $c = -2$, the $SL(2)$ vacuum coincides with the ghost ground state enabling the application of the replica formulation, which yields negative entanglement entropy. This also involves appropriate inner products for states necessary for nonvanishing correlation functions. Similar discussions apply for a CFT of anti-commuting scalars which is a logarithmic CFT. We have also discussed a toy model of two “ghost-spins” with non-positive inner products mimicking some of these features, where the reduced density matrix gives von Neumann entropy for negative norm states with a negative real part as well as a constant imaginary part.

Our analysis here illustrates that negative EE can formally be obtained from appropriate generalizations of the standard CFT replica technique in toy 2-dim ghost CFTs with negative central charge: this is motivated by the 3-dim $Sp(N)$ higher spin theory in [12] but the eventual results appear independent of dS/CFT per se. Towards regarding the areas of dS_4 complex extremal surfaces in [19, 20] as entanglement entropy in the dual Euclidean CFT_3 with negative central charge, it would be interesting to understand if these features of

negative EE in the 2-dim toy investigations here carry over in the sense here to that case³. It is worth noting however that while the CFTs discussed here do not exhibit any obstruction to a Lorentzian continuation, the dual CFT in the de Sitter context is intrinsically Euclidean: the boundary Euclidean time direction implicit in the entanglement calculations is simply one of the boundary spatial directions along which there is translation invariance (see *e.g.* [13] for discussions on a state-operator correspondence in the dS/CFT context). Relatedly in dS_3 , the dual is expected to be a 2-dim CFT with imaginary central charge, with likely new features altogether (see *e.g.* [9, 10] for aspects of novel hermiticity relations): the CFTs here with negative central charge are not to be considered as dual to dS_3 .

It is worth emphasising that the analysis here is formal, suggesting that a formal generalization of the usual notions of entanglement entropy for unitary theories can be defined for certain ghost CFTs with negative central charge. The resulting quantity is negative, as are the areas of the dS_4 complex extremal surfaces reviewed in sec. 2. Such a negative entanglement entropy does not satisfy properties of ordinary EE such as strong subadditivity: it has various odd features, and the physical interpretation is far from clear. Indeed we call this entanglement entropy only in the sense of an extension of the usual techniques to this case. Let us recall the area (4) of the codim-2 complex extremal surfaces in dS_4 for a strip (and sphere) subregion on a constant boundary Euclidean time slice studied in [19, 20]. We have $S_A \sim -\frac{R_{dS}^2}{G_4}(\frac{V_1}{\epsilon} - \frac{V_1}{l})$ for a strip of finite width l . Then, as discussed in [19], the analog of mutual information for two disjoint strip subregions A, B defined as $I[A, B] = S[A] + S[B] - S[A \cup B]$ is negative definite for A, B sufficiently nearby (and vanishes beyond a critical separation)⁴. (The negative sign suggests that $-S[A]$ satisfies strong subadditivity.) Secondly, consider two strip subregions of width l_2 and $l_1 > l_2$ (for intervals with finite widths $l_1, l_2 \ll V_1$)⁵. Then we see that $S(l_1) - S(l_2) = -\frac{R_{dS}^2}{G_4}(\frac{V_1}{l_2} - \frac{V_1}{l_1}) < 0$, *i.e.* $S(l_1) < S(l_2)$. Formally the $l \rightarrow \infty$ limit⁶ gives $S(l) \rightarrow -\infty$ so that any finite l gives $S(l) > S(\infty)$: relatedly as $l \rightarrow \epsilon$, we have $S(l) \rightarrow 0^-$. This means that a bigger subregion is more ordered than a smaller one, in contrast with a conventional unitary CFT where

³Compactifying a free massless 3-dim CFT along one direction gives several 2-dim massive theories: naively the entanglement entropy summing over modes is estimated as (104) using the 2-dim CFT results here, and assuming (101) can be established for negative central charge as well.

⁴Entanglement entropy for a single interval involves the 2-point function of twist operators in the CFT. Multiple intervals may be more interesting: for instance, mutual information for two disjoint intervals A and B involves the 4-point function of twist operators with more intricate dependence on the full CFT structure. Relatedly it would be interesting to study excited states.

⁵In the CFT context here, we have implicitly assumed that the single interval A in question here is finite (although it can be large). The presence of the background charge makes it difficult to address the question of S_A matching the entanglement entropy of the complement of the subsystem in this context.

⁶It is interesting to ask if there are analogs in the bulk de Sitter space context of the background charge appearing in our CFT discussions here.

$S(l_1) > S(l_2)$ *i.e.* a bigger subregion is more disordered than a smaller one. Thirdly, the entropic c-function [54, 55] defined as $c(l) = l \frac{dS_A}{dl}$ for unitary 2-dim theories with positive central charge shows $c(l)$ to be monotonically decreasing as the size l increases, with $c'(l) \leq 0$. In higher dimensions, this can be defined as $c(l) = \frac{l^{d-1}}{V_{d-2}} \frac{dS_A}{dl}$ for strip-shaped subregions, with V_{d-2} the interface area (see *e.g.* [24]). The finite cutoff-independent part of entanglement entropy in this regard encodes useful information about the flow towards long wavelengths. Applying this to the dS_4 complex surfaces area, we have $c(l) \equiv \frac{l^2}{V_1} \frac{dS_A}{dl} = -\frac{R_{dS}^2}{G_4} < 0$ which is negative, reflecting the comments above, *i.e.* as l increases, $S(l)$ decreases. For more general asymptotically dS_4 spaces, the areas S_A of corresponding complex extremal surfaces would again appear to be negative (as noted in [19] for dS_4 black branes [18]). This would imply that $c'(l) > 0$, *i.e.* as the size l increases, $c(l)$ will increase. This suggests that new degrees of freedom are *integrated in*. This is a little reminiscent of the picture in [8] of time evolution encoded as inverse RG flow. In the present context, recall that these complex extremal surfaces were obtained along the path $\tau = iT$, the turning point being $\tau_* = il$: however each such complex τ can be mapped to a corresponding real-valued τ location in the bulk dS_4 space (as discussed in sec. 2), with $|\tau_*| = l$. Thus increasing size l corresponds to going to larger $|\tau_*|$, *i.e.* earlier times in the past. It would be interesting to explore this.

Finally, the dS/CFT dictionary $Z_{CFT} = \Psi_{dS}$ suggests that, unlike in AdS/CFT , entanglement entropy in the dual CFT here is not the entanglement entropy of bulk fields in dS (see *e.g.* [56]). The CFT entanglement entropy, in a replica formulation, involves $\frac{Z_n}{Z_1^n}$ which naively maps to $\frac{\Psi_n}{\Psi_1^n}$: relatedly, the bulk density matrix involves $\Psi^* \Psi \sim Z^* Z$ which is expected to lead to very different entanglement structures. Thus while the areas of the de Sitter complex extremal surfaces in [19, 20] might be akin to entanglement entropy in the dual CFT, the bulk interpretation of these would be interesting to understand better, perhaps along the lines of [26, 27, 28].

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A 2-dim CFT and the replica formulation

Here we review some aspects of the replica formulation of entanglement entropy [32, 33], for a 2-dim Euclidean CFT with central charge c , keeping in mind our context in sec. 3.2. Consider a subsystem A comprising a single interval between $x = u$ and $x = v > u$ on a time slice $t_E = \text{const.}$ The path integral for n copies of the subsystem is $\text{tr} \rho_A^n = \prod_{k=1}^n \langle \Phi_k^u \Phi_k^v \rangle$ where the boundary conditions joining each copy k to its adjacent next copy in the replica space are implemented by twist operators Φ_k^u at $w = u$ and Φ_k^v at $w = v$. The conformal dimensions of these twist operators Φ_k^u, Φ_k^v can be obtained as in [32, 33] by using the conformal map

$$z = \left(\frac{w - u}{w - v} \right)^{1/n} \quad (96)$$

from the replica w -space which contains a conical singularity around each twist field location, to the replica z -space \mathbb{C} . We note that the conformal map $w \rightarrow \frac{w-u}{w-v}$ maps the interval points $w = u$ and $w = v$ to $0, \infty$ respectively: the z transformation then maps the n w -sheets to a single complex plane \mathbb{C} . Near $w \sim u$ and $w \sim v$, we have conical singularities $z \sim (w - u)^{1/n}$ and $z \sim (w - v)^{-1/n}$ so that the neighbourhood of these points resembles $z = w^{\pm 1/n}$. The metric near these locations is $dzd\bar{z} = |\partial_w z|^2 dw d\bar{w} = \frac{|w|^{-2+2/n}}{n^2} dw d\bar{w}$. Under a conformal transformation $w \rightarrow z$, the energy-momentum tensor transforms as (31). The condition (32), *i.e.* $\langle T(z) \rangle_{\mathbb{C}} = 0$, is equivalent to taking the z -plane to represent the CFT ground state, and so by the state-operator correspondence, equivalent to the insertion of the identity operator which corresponds to the ground state. Now taking expectation values (with \mathcal{R}_n the n -sheeted w -space), we obtain (33). For u, v well-separated, *e.g.* $v \gg u$, the expression (33) simplifies to $\langle T(w) \rangle_{\mathcal{R}_n} \sim \frac{c(1-\frac{1}{n^2})}{24} \frac{1}{(w-u)^2}$ which corresponds to a conformal transformation $z = (w - u)^{1/n}$ showing a conical singularity at $w \sim u$. The boundary conditions at u, v are equivalent to the insertion of twist operators $\Phi_n(u), \Phi_{-n}(v)$ at $w = u, v$ respectively. The last expression in (33) has been written after comparing with the standard expression $\langle T(z) \phi_1(w_1) \phi_2(w_2) \rangle = \sum_i \left(\frac{h_i}{(z-w_i)^2} + \frac{1}{(z-w_i)} \frac{\partial}{\partial w_i} \right) \langle \phi_1(w_1) \phi_2(w_2) \rangle$ for the 3-point function of $T(w)$ with two primary operators, *i.e.*

$$\begin{aligned} \langle T(w) \Phi_n(u) \Phi_{-n}(v) \rangle &= \frac{\Delta_n}{(w-u)^2 (w-v)^2 (v-u)^{2\Delta_n-2} (\bar{v}-\bar{u})^{2\bar{\Delta}_n}} \\ &= \left(\frac{\Delta_n}{(w-u)^2} + \frac{\Delta_n}{(w-u)^2} + \frac{1}{(w-u)} \frac{\partial}{\partial u} + \frac{1}{(w-v)} \frac{\partial}{\partial v} \right) \langle \Phi_n(u) \Phi_{-n}(v) \rangle. \end{aligned} \quad (97)$$

(We have effectively assumed that w here represents a single sheet of the n -sheeted w -space.) Comparing the second line with (33) leads to the correlation function of the twist operators,

$$\langle \Phi_n(u) \Phi_{-n}(v) \rangle = |v - u|^{-2\Delta_n - 2\bar{\Delta}_n}, \quad \Delta_n = \frac{c(1-\frac{1}{n^2})}{24} = \bar{\Delta}_n. \quad (98)$$

Writing (33) as $\langle T(w) \rangle = \frac{\int D\varphi T(w) e^{-S}}{\int D\varphi e^{-S}}$ we can write the partition function of the CFT on the n -sheeted replica space with these boundary conditions as

$$\text{tr} \rho_A^n = \prod_{k=1}^n \langle \Phi_k(u) \Phi_{-k}(v) \rangle = |v - u|^{c(n - \frac{1}{n})/6}, \quad (99)$$

so that the entanglement entropy for the interval is

$$S_A^{EE} = -\lim_{n \rightarrow 1} \partial_n \text{tr} \rho_A^n = \frac{c}{3} \log \frac{l}{\epsilon}, \quad (100)$$

where $l \equiv |v - u|$ is the length of the interval and ϵ is an ultraviolet cutoff.

The replica formulation above for entanglement entropy appears to require no input besides the central charge of the CFT: however there are various implicit assumptions where unitarity has entered. In sec. 3.2, we have discussed these and corresponding modifications in the context of the bc -ghost system.

Higher dimensional free fields compactified to 2-dim: Considering first a massive 2d theory, we imagine the mass m is the only scale in the system, which plays the role of the correlation length $\xi \sim \frac{1}{m}$. Revisiting the discussion [32] for entanglement entropy for a single interval (for simplicity semi-infinite) in this massive theory gives

$$S_A = -\frac{c}{6} \log(m\epsilon). \quad (101)$$

Now following [22], we consider a d -dimensional Euclidean CFT comprising massless free fields. Compactifying this on a $(d - 2)$ -dim torus T^{d-2} gives an effective 2-dim theory on \mathbb{R}^2 with many massive fields of mass

$$m^2 = \sum_{i=3}^d k_i^2 = \left(\frac{2\pi}{L}\right)^2 \sum_{i=3}^d n_i^2, \quad k_i = \frac{2\pi n_i}{L}. \quad (102)$$

The correlation length is $\xi \sim \frac{1}{m}$ for modes of mass m . We are considering a strip subsystem with width l along the noncompact direction, and stretched along the other directions which we compactify. Then the entanglement entropy for this subsystem can be estimated by summing over all these modes using the results for $l \ll \xi$ and $l \gg \xi$ becomes

$$\begin{aligned} S_A &= \sum_{k_3, k_4, \dots, k_d}^{\xi < l} \frac{c}{3} \log \frac{\xi}{\epsilon} + \sum_{k_3, k_4, \dots, k_d}^{\xi \geq l} \frac{c}{3} \log \frac{l}{\epsilon} \\ &= \frac{c}{3} \left(\frac{L}{2\pi}\right)^{d-2} \left[\int_{1/l}^{1/\epsilon} d^{d-2} k (-\log(|k|\epsilon)) + \int_0^{1/l} d^{d-2} k \log \frac{l}{\epsilon} \right]. \end{aligned} \quad (103)$$

Noting $\int k^{d-3} dk \log k = \frac{k^{d-2}}{d-2} (\log k - \frac{1}{d-2})$, the above expression shows various cancellations and simplifies as

$$S_A = \frac{c}{3} \frac{\Omega_{d-2}}{(2\pi)^{d-2} (d-2)} \left(\frac{L^{d-2}}{\epsilon^{d-2}} - \frac{L^{d-2}}{l^{d-2}} \right). \quad (104)$$

The expression (104) for entanglement entropy obtained by compactifying a higher dimensional CFT to 2-dim (using (101)) naively appears to be valid beyond the realm of conventional unitary CFTs. In particular for negative central charge, (104) has the same form as the area (4) of the dS_4 complex extremal surfaces we reviewed earlier. In this context, we imagine taking a free 3-dim Euclidean CFT of negative central charge $\mathcal{C} \sim -\frac{R_{dS}^2}{G_4}$ (obtained from the holographic $\langle TT \rangle$ correlators) to be compactified along one direction giving several 2-dim massive theories. This is simply a heuristic argument to obtain a rough estimate for the form of entanglement entropy in a higher dimensional CFT.

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